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MATCHED ASYMPTOTIC EXPANSIONS OF GLOBAL HARMONIC SOLUTIONS OF THE DUFFING EQUATION*

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MATCHED ASYMPTOTIC EXPANSIONS OF

GLOBAL HARMONIC SOLUTIONS OF THE DUFFING EQUATION

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Abstract

Harmonic solutions of the inhomogenous,damping free,Duffing equation are found using a variant of the method of matched asymptotic expansions due to B.Matkowsky and E. Reiss,[3].These solutions,which have the frequency ω of the forcing term,are uniformly valid in ω and close to the solutions of the bifurcation problem determined by the homogeneous Duffing equation.

1. Introduction

This paper is concerned with the harmonic solutions of the Duffing equation without damping:

(1.1) $$\tilde{G}[x,\omega,\delta] = \frac{d^2 x}{d^2\tau} + x - \frac{1}{3}x^3 - \delta\cos\omega\tau = 0 \qquad |\delta| << 1$$

The same problem has been the subject of several previous investigations,[2],[6].The novelty of the present analysis consists in the method applied,which yields global asymptotic expansions of periodic solutions of (1.1) with period $2\pi/\omega$.

We seek twice continuously differentiable solutions $x(t,\delta)$ of (1.1) satisfying

$$x(\tau,\delta) = x(\tau + 2\pi/\omega,\delta),$$

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We define the new time variable t by

$$t = \omega\tau,$$

we then express (1.1) as

(1.2) $$G[x,\omega,\delta] = \omega^2 x'' + x - \frac{1}{3}x^3 - \delta\cos t = 0,$$

$$(\ )'' = \frac{d^2}{dt^2}.$$

Thus we seek solutions of (1.2) satisfying:

(1.3) $$x(t,\delta) = x(t + 2\pi,\delta).$$

From now on,we shall refer to equations (1.2), (1.3) as the <u>perturbed problem</u>

We consider G as an operator on the linear manifold of $L^2[0,2\pi]$ of the twice continuously differentiable, 2π - periodic,real valued functions x(t),with $x'(0)=0$. We normalize the inner product in $L^2[0,2\pi]$ by

$$< x;y > = \frac{1}{\pi}\int_0^{2\pi} x(t)y(t)dt$$

We consider the harmonic solutions of the <u>bifurcation problem</u>

(1.4) $$G[x,\omega,0] = 0,$$

(1.5) $$x(t) = x_0(t) = 0\ ,\ \omega \text{ arbitrary}$$

and

$$(1.6) \qquad x(t) = x_1(t) = \varepsilon \cos t - \frac{\varepsilon^3}{96}\cos 3t + O(\varepsilon^4),$$

provided

$$(1.7) \qquad \omega = \omega(\varepsilon) = 1 - \frac{1}{8}\varepsilon^2 + O(\varepsilon^3).$$

The small parameter ε is the amplitude of the solution of (1.4), i.e.,

$$|x|^2 = \frac{1}{\pi}\int_0^{2\pi} x^2(t)\, dt = \varepsilon^2$$

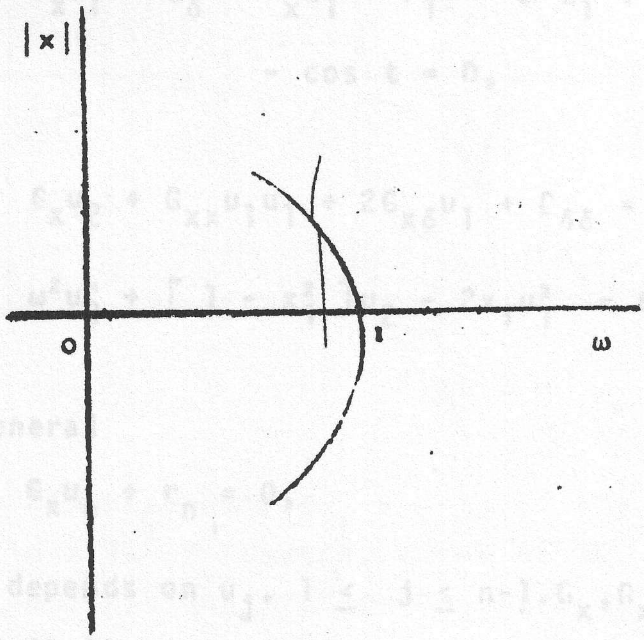

Fig. 1 Response curves of the solutions of the bifurcation problem (1.4).

2. Outer Expansions

In order to find asymptotic solutions of the perturbed problem, uniformily valid in ω and close to the solutions (1.5) and (1.6),(1.7),we apply the method of singular perturbations of bifurcations,[3].

We begin by looking for outer solutions of (1.2), (solutions for ω bounded away from $\omega_o = 1$),in the form:

(2.1) $$u(x_i) = u(t,x_i,\omega,\delta) = x_i(t) + \sum_{j=1}^{\infty} u_j \delta^j \quad ,(x_i, i=0,1 \text{ defined in } (1.5), (1.6))$$

Substituting x in (1.2) by u in (2.1),one obtains the recursive system of linear equations:

(2.2) $$G_x u_1 + G_\delta = G_x u_1 + r_1 = \omega^2 u_1'' + [1 - x_i^2] u_1 - \cos t = 0,$$

(2.3) $$G_x u_2 + G_{xx} u_1 u_1 + 2G_{x\delta} u_1 + G_{\delta\delta} = G_x u_2 + r_2 = \omega^2 u_2'' + [1 - x_i^2] u_2 - 2x_i u_1^2 = 0,$$

and in general

$$G_x u_n + r_n = 0,$$

where r_n depends on u_j, $1 \leq j \leq n-1$. $G_x, G_{xx}, G_{\delta\delta}$ etc. in (2.2) and (2.3) are evaluated at $u = x_i$, $\delta = 0$

When $\omega = \omega_o = 1$,the operator

$$G_x^o = G_x [0,1,0]$$

is not invertible because it corresponds to the variational problem of the bifurcation problem (1.4).Since the inhomogeneous term $G_\delta = -\cos t$ in (2.2) is not orthogonal to the kernel of the adjoint operator $(G_x^o)^+$,expansions of the form (2.1) can not be valid as $\omega \to \omega_0$

When i = 0 in (2.1), equation (2.2) is reduced to

$$\omega^2 u_1'' + u_1 - \cos t = 0,$$

and therefore

$$u(x_0) = u(t, x_0, \omega, \delta) = \frac{\delta}{1-\omega^2}\cos t + O(\delta^2). \tag{2.4}$$

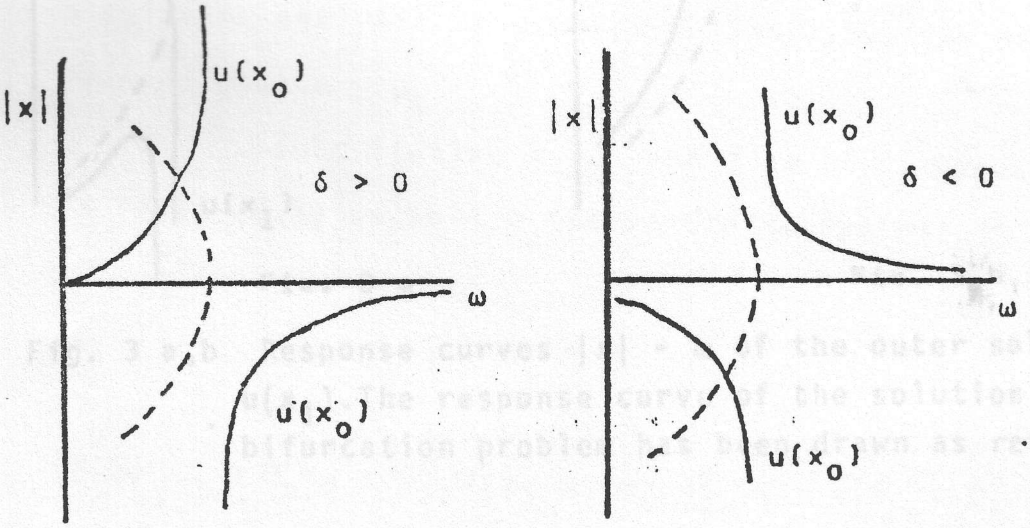

Fig. 2 a   Fig. 2 b

Fig. 2 a,b Response curves $|x| - \omega$ of the outer solution $u(x_0)$. The response curve of the solution of the bifurcation problem has been drawn as reference.

When i = 1 in (2.1), equation (2.2) can be solved by expansions in the eigenfunctions of C_x, i.e., in the solutions of

$$\omega^2(\varepsilon)\phi'' + [1 - x_1^2(\varepsilon)]\phi = \lambda\phi.$$

This yields:

$$u(x_1) = u(t,x_1,\omega,\delta) = x_1(t,\varepsilon) - \frac{2\delta}{\varepsilon^2}\cos t + O(\varepsilon^2,1 \qquad (2.5)$$

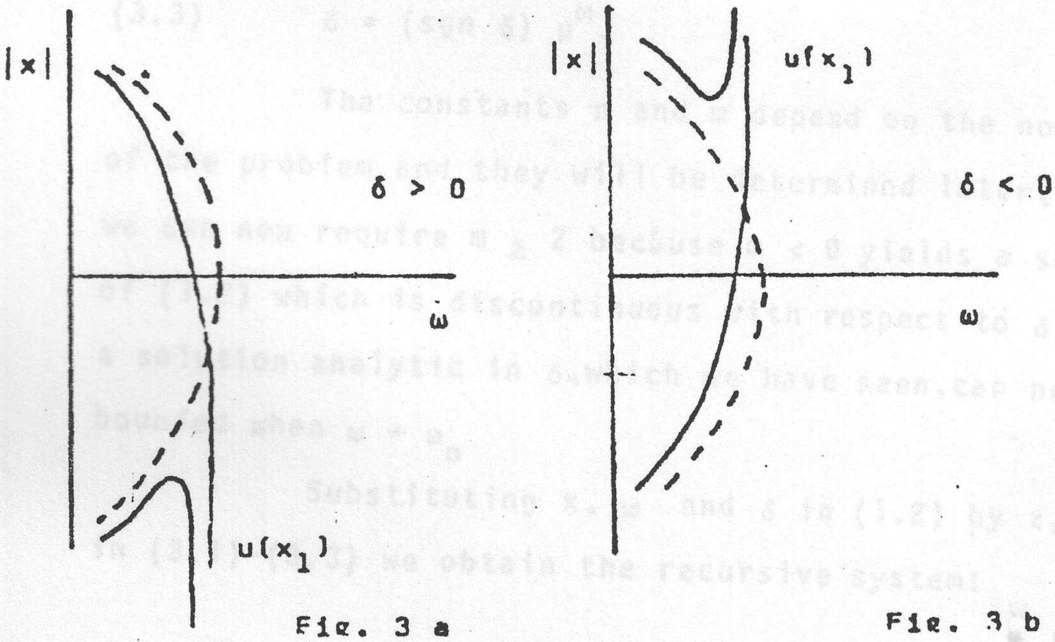

Fig. 3 a Fig. 3 b

Fig. 3 a,b Response curves $|x| - \omega$ of the outer solution $u(x_1)$. The response curve of the solution of the bifurcation problem has been drawn as reference.

3. Inner Expansions

Inner solutions of (1.2), i.e., bounded solutions valid for ω close to ω_o, are found stretching the neighborhoods of ω. Thus we seek parametrized solutions of (1.2) in the form

$$Z(t,\mu) = \sum_{j=1}^{\infty} z_j(t)\mu^j, \qquad (3.1)$$

with

$$\omega = \omega_o + \xi\mu^n + \sum_{j=2}^{\infty} \xi_j(\mu^n)^j \qquad (3.2)$$

The small parameter μ in (3.1) and (3.2) is defined by

$$\delta = (\text{sgn}\ \delta)\ \mu^m. \tag{3.3}$$

The constants n and m depend on the nonlinearity of the problem and they will be determined later, however, we can now require $m \geq 2$ because $m < 0$ yields a solution of (1.2) which is discontinuous with respect to δ, and $m = 1$ a solution analytic in δ, which we have seen, can not be bounded when $\omega \to \omega_0$

Substituting x, ω and δ in (1.2) by z, ω and δ in (3.1)-(3.3) we obtain the recursive system:

$$G_x^o z_1 = z_1'' + z_1 = 0 \tag{3.4}$$

$$G_x^o z_2 + G_{xx}^o z_1 z_1 + 2\omega_\mu\ G_{x\omega}^o z_1 + G_\delta^o \delta_{\mu\mu} = G_x^o z_2 + R_2 = \tag{3.5}$$

$$z_2'' + z_2 + 4\omega_\mu z_1'' - \delta_{\mu\mu} \cos\ t = 0$$

$$G_x^o z_3 + G_{xxx}^o z_1 z_1 z_1 + 3\{G_{xx}^o z_1 z_2 + \omega_\mu [G_{x\omega}^o z_2 \tag{3.6}$$

$$+ G_{xx\omega}^o z_1 z_1] + \omega_\mu^2 G_{x\omega\omega}^o z_1 + \omega_{\mu\mu} G_{x\omega}^o z_1 + \delta_{\mu\mu} G_x^o z_1\} +$$

$$\delta_{\mu\mu\mu} G_\delta^o = G_x^o z_3 + R_3$$

$$= z_3'' + z_3 + 6\omega_\mu z_2'' + 6\omega_\mu^2 z_1'' + 6\omega_{\mu\mu} z''_1$$

$$- 2z_1^3 - \delta_{\mu\mu\mu} \cos\ t = 0$$

and in general

$$G_x^o z_n + R_n = 0.$$

$( )_{,\mu} = \frac{d}{d\mu}$, $( )^o$ means evaluation at $x = 0, \omega = 1, \delta = 0$

In (3.4)-(3.6) we have used the fact that $G[0,\omega,0] = 0$ for every ω and the requirement $\delta_\mu = 0$

Let

$$z_1 = A \cos t$$

be the solution of (3.4), then, the solvability condition for (3.5) is

$$4\omega_\mu A + \delta_{\mu\mu} = 0$$

If $\omega_\mu \neq 0$, then, $A = -\delta_{\mu\mu}/4\xi$ which is either unbounded as $\xi \to 0, (\omega \to \omega_o)$, or zero. In order to obtain a solution which depends on both ω and δ, we require $\omega_\mu = \delta_{\mu\mu} = 0$. Hence,

$$z_2 = B \cos t$$

The solvability condition for (3.6) is

(3.7) $$A^3 + 8\xi A + 4 \operatorname{sgn} \delta = 0,$$

determining $n = 2, m = 3$.

Let

$$\xi_c = -3(128)^{-1/3}$$

Equation (3.7) posseses three real solutions if $\xi < \xi_c$ and only one if $\xi > \xi_c$.

These solutions are:

$$A_1^-(\xi) = 2\sqrt{(-8\xi/3)}\cos\theta$$
$$A_2(\xi) = 2\sqrt{(-8\xi/3)}\cos(\theta + 2\pi/3) \qquad \xi < \xi_c$$
$$A_3(\xi) = 2\sqrt{(-8\xi/3)}\cos(\theta + 4\pi/3)$$

where

$$\cos(3\theta) = -\ 2\,\mathrm{sgn}\,\delta\sqrt{(-3/8\xi)^3}$$

and

$$A_1^+(\xi) = -[2\,\mathrm{sgn}\,\delta - \sqrt{\Delta}]^{1/3} - [2\,\mathrm{sgn}\,\delta + \sqrt{\Delta}]^{1/3} \qquad \xi > \xi_c$$

with

$$\Delta = 4 + (8\xi/3)^3$$

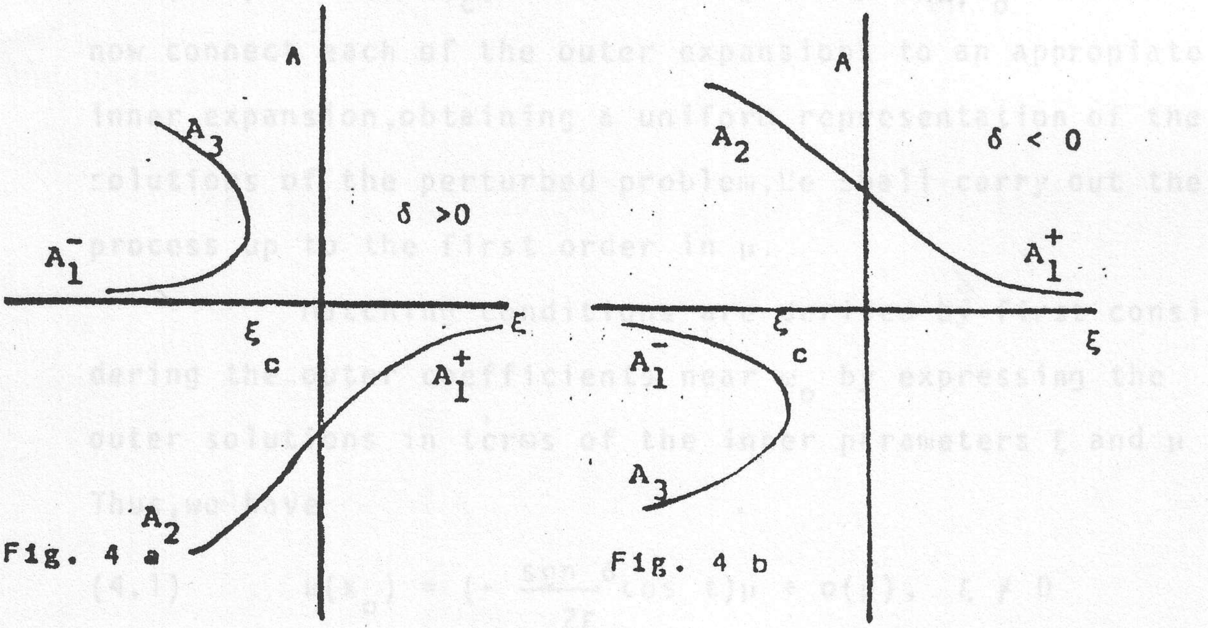

Fig. 4 a   Fig. 4 b

Fig. 4 a,b Solutions of equation (2.7)

Accordingly, we have the following four solutions of (3.4)

$$(3.8) \qquad \left.\begin{aligned} z_1^{1-}(t,\xi) &= A_1^-(\xi)\cos t \\ z_1^{j}(t,\xi) &= A_j(\xi)\cos t\;;\; j=1,2 \end{aligned}\right\}\quad \xi<\xi_c$$

$$(3.9) \qquad z_1^{1+}(t,\xi) = A_1^+(\xi)\cos t \qquad \xi>\xi_c$$

4. Matching

We have obtained the outer expansions (2.4) and (2.5) of the solutions of the perturbed problem (1.2), corresponding to each one of the solution branches (1.5) and (1.6),(1.7) of the bifurcation problem. In addition, we have determined the four inner solutions, (3.8) for $\xi < \xi_c$ and (3.9) for $\xi > \xi_c$, near the singular point ω_o. We shall now connect each of the outer expansions to an appropiate inner expansion, obtaining a uniform representation of the solutions of the perturbed problem. We shall carry out the process up to the first order in μ.

Matching conditions are derived by first considering the outer coefficients near ω_o by expressing the outer solutions in terms of the inner parameters ξ and μ Thus, we have

$$(4.1) \qquad u(x_o) = (-\frac{\operatorname{sgn}\delta}{2\xi}\cos t)\mu + o(\mu), \quad \xi \neq 0$$

$$(4.2) \qquad u(x_1) = [(\pm\sqrt{-8\xi} + \frac{\operatorname{sgn}\delta}{4\xi})\cos t]\mu + o(\mu), \quad \xi<\xi_c$$

These outer expansions are valid for ω bounded away from ω_o, the inner expansions (3.8),(3.9) are valid for ω near ω_o. We assume, as it is customary in the theory in the theory of matched asymptotic expansions, that there is a common interval in which both the inner and outer expansions are valid and that this interval shrinks to ω_0 when $\delta \to 0$. Since the inner and outer expansions are asymptotic expansions of the same function, their difference must be asymptotic to zero on the common interval, then, the limit as $|\xi| \to \infty$ of each coefficient of μ in

$$u(t,x_i(t,\mu),\omega(\mu),\delta(\mu)) - Z(t,\mu)$$

must be zero.

Observing that as $\xi \to -\infty$,

$$\theta - \pi/2 - \frac{4}{3}(-3/8\xi)^{3/2}$$

and when $\xi \to +\infty$,

$$(2\ \mathrm{sgn} \pm \sqrt{\Delta}\ )^{1/3} - \pm\left(\frac{8\xi}{3}\right)^{1/2} + \frac{2}{3}\ \mathrm{sgn}\ \delta\left(\frac{8\xi}{3}\right)^{-1} \pm \frac{2}{3}\left(\frac{8\xi}{3}\right)^{-5/2}$$

we have

$$A_1^-(\xi) -- \frac{\mathrm{sgn}\ \delta}{2\xi}$$

$$A_2(\xi) -- \sqrt{-8\xi}$$

$$A_3(\xi) - \sqrt{-8\xi}$$

and

$$A_1^+(\xi) -- \frac{\mathrm{sgn}\ \delta}{2\xi} .$$

Therefore, we obtain the connections described in the following diagram

	$u(x_o)$	Z^{1-}
$\xi < \xi_c$	$u(x_1)$ $\varepsilon < 0$	Z^2
	$u(x_1)$ $\varepsilon > 0$	Z^3
$\xi > \xi_c$	$u(x_o)$	Z^{1+}

The response curve $|x| - \omega$ of the composite solutions, Fig.5 a,b, is equal to those obtained by other methods, (see for example, [6] ,page 88).

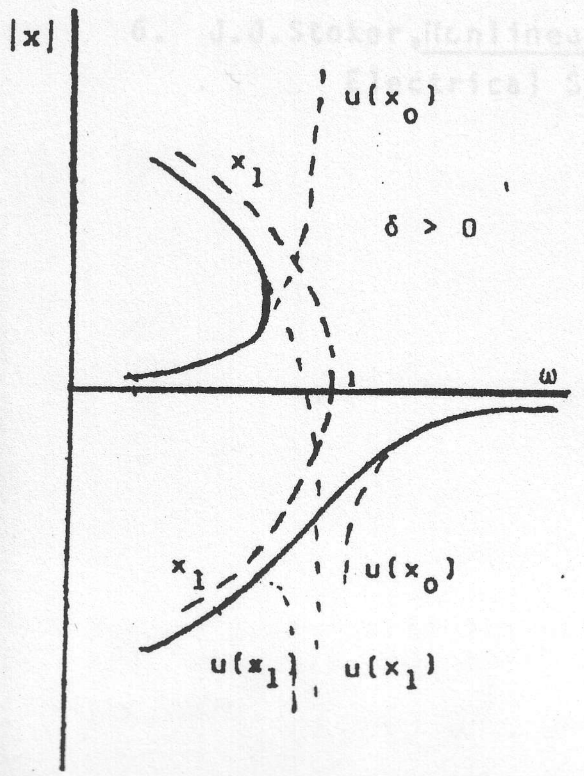

Fig. 5 a

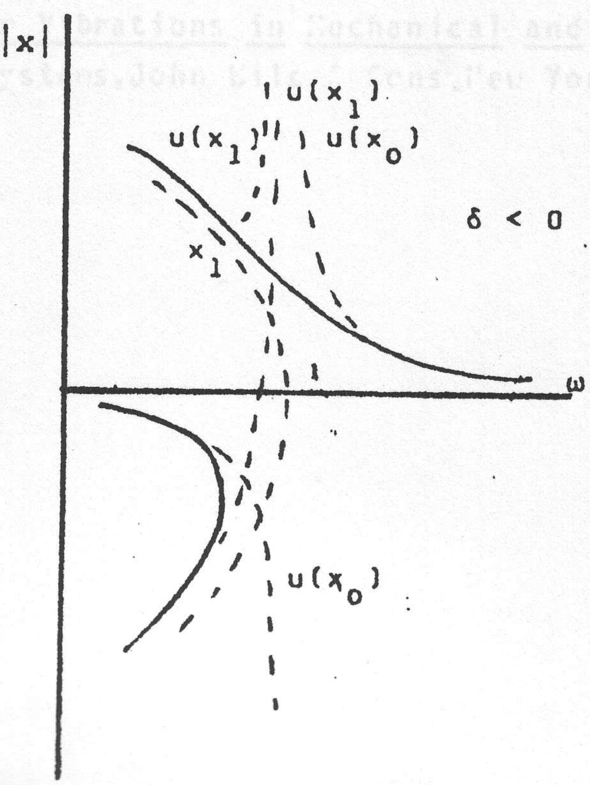

Fig. 5 b

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