REGULAR METRIC
DEFINITION AND CHARACTERIZATION IN THE DISCRETE PLANE

(talk - transparencies)

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Cover
In this talk, we will introduce a new concept that we call “regular metric” and we will show a possible characterization of it in the discrete plane.

Content
In order to motivate our work, we will show the loss of a geometrical property in the discrete plane;
then we will give the axiomatic definition of regular metric space and two equivalent simpler definitions;
following the definition, we will show that the regularity is a sufficient condition for some important geometrical properties;
finally we show a possible characterization in the discrete plane

Loss of a geometrical property in the discrete plane
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In the left part of the figure we see a circle and a straight–line. Both are objects derived from the Euclidian Metric defined on the continuous plane. Since the straight–line goes through the circle center, these two objects intersect each other. Now, if we restrict the Euclidian Metric to the discrete plane, we see that the circle reduces to these four points and the straight–line reduces to these five other points and we observe that the two objects have no more intersection. We have lost an important geometrical property. In the right part of the figure we present another similar example based on a elliptic metric.
This is our motivation to introduce a new concept that we call “regular metric”.

Axiomatic definition

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We first introduce an axiomatic definition. This definition is based on three axioms because the straight–line in the metric sense can be decomposed into three parts. Axiom 1 says that, given any points *x* and *y*, every circle of center *x* and radius greater than the distance between the points *x* and *y* intersects part 1 of the straight–line passing through these points. We call this property: lower regularity of type 1. Actually, it is equivalent to the lower regularity for the triangle inequality introduced by Kiselman in 2002.

Axiomatic definition

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Axiom 2 says that every circle of center *x* and radius less than the distance between the points *x* and *y* intersects part 2 of the straight–line passing through these points. We call this property: lower regularity of type 2.

Axiomatic definition

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Axiom 3 says that every circle of center *x* intersects part 3 of the straight–line passing through the points *x* and *y*. We call this property: upper regularity. Actually, it is equivalent to the upper regularity for the triangle inequality introduced by Kiselman in 2002.

Axiomatic definition

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A metric is space is regular if the previous three axioms are satisfied.
The chessboard and city block distances are example regular metric.

Equivalent definition

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We now present an equivalent definition of regular metric space. We say that a metric space is *regular* if every straight–line passing through the center of every circle has at least two diametrically opposite points.
The figure displays two points *x* and *y*, the straight–line with respect to the Chessboard distance, passing through *x* and *y*, and the circle of center *x*, and radius 2. In the case of the Chessboard distance (which is regular), we can always find two diametrically opposite points, for example the points *u* and *v* on the right part of the figure.

Some properties

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The first property guarantees that the classical sufficient condition to have intersection between two balls, works in a lower metric space of type 2. This is not always true for every metric. Two counter examples show that not always two balls intersect each other even though the distance between their centers are less than the sum of their radii. In the first counter example the elliptic distance between the centers is less than 2 and the sum of the ball radii is 2 nevertheless the two balls don't intercept each other.

Some properties

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In the second counter example, the octagonal distance between the centers is equal to the sum of the ball radii but the two balls don't intercept each other. The figure contains two different such situation. The second one is due to Rosenfeld in 1968.

Some properties

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The second property guarantees that the classical necessary condition to have ball inclusion, works in a upper metric space. A counter example shows that not always the distance between the centers of two balls is less than or equal to the difference between the radii of the two balls even though one is included in the other. In the counter example one ball is included in the other but the distance between their centers is 2 and is greater than the difference of their radii which is 2 minus 1, that is 1.

Characterization

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Let us introduce the elements for the characterization.

The first element is the recursive Minkowski sum of a nonempty subset times a positive integer. For j greater than 1, jB is the Minkowski sum of $(j - 1)B$ plus B.

The second element is the closure property. By definition, a subset B has the closure property if the subsets jB are invariant of the morphological closure by B.

The third element is the induced metric; given a symmetric subset, the induced distance between two points is given by the least integer j such that the difference of the two points is still in jB.

Finally the last element is the classical unit ball of a metric, centered at the origin.

Characterization

2/2

The characterization theorem says that the set of regular metrics defined on the discrete plane that are onto the set of natural numbers is in relation one to one with the set of symmetric subsets of the discrete plane which have the closure property. The chessboard distance is an example of such metric. It maps to the three by three square and can be reconstructed from it by using the recursive Minkowski addition.

From this result, we see that the metric spaces whose balls cannot be obtained from the unit ball through the recursive Minkowski addition are not regular. A simple example is the octagonal distance which is consequently not regular.

Future work

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An interesting question for future work, would be to find a sufficient condition for a subset B to have the closure property.

References

A detail study of the present subject can be found in an on–line Technical Note at INPE Digital Library.