0237

NEW PERTURBATION SOLUTIONS OF THE NONLINEAR SHALLOW-WATER EQUATIONS IN THE β-PLANE

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Abstract. In the present work we find basic solutions of the β-plane approximation of the equations of motion of a uniform, inviscid fluid relative to rotating axis in a region of a spherical surface. We also apply a perturbation scheme to obtain a dispersion relation which defines the existence of travelling-wave solutions of the equations.

1. Introduction. When the problem of motion of a uniform inviscid fluid relative to rotating axes is considered in a region of a spherical surface such that its characteristic lengths are much smaller than the radius of the sphere, the equation of motion is approximately reduced to that of a two-dimensional flow in a plane layer of fluid. If, furthermore, the relative vorticity is small compared with the Coriolis parameter C then making the approximation

$$C = C_0 + \beta y$$

it can be proved that the equations that describe the problem[1] are

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} + \tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}} - \beta\tilde{y}\tilde{v} = -\frac{\partial \tilde{\Phi}}{\partial \tilde{x}},$$

$$\frac{\partial \tilde{v}}{\partial \tilde{t}} + \tilde{u}\frac{\partial v}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{v}}{\partial \tilde{y}} + \beta\tilde{y}\tilde{u} = -\frac{\partial \tilde{\Phi}}{\partial \tilde{y}},$$

$$\frac{\partial \tilde{\Phi}}{\partial \tilde{t}} + \tilde{u}\frac{\partial \Phi}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{\Phi}}{\partial \tilde{y}} + \tilde{\Phi}\left(\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}}\right) = 0.$$

called the β-plane approximation.

In a recent work[2], Raupp and Loula proved the existence of travelling wave solutions of (1). The purpose of the present work is to efectively calculate such solutions using an adequate perturbational scheme which explores the quadratic character of the non-linearities

Let $\bar{H}$ be a reference, constant geopotential, then if we define

$$\tilde{u} = \sqrt{\bar{H}}u; \qquad \tilde{v} = \sqrt{\bar{H}}v; \qquad \tilde{\Phi} = \bar{H}\Phi;$$

(2)

$$\tilde{x} = \left(\frac{H}{\beta^2}\right)^{1/4} x; \quad \tilde{y} = \left(\frac{H}{\beta^2}\right)^{1/4} y; \quad \tilde{t} = \left(\bar{H}\beta^2\right)^{1/4} t.$$

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and introduce the new variables in (1) and cancelling common factors, the resulting equations are parameter-free, or, equivalently, they are equal to (1) with $\beta = 1$ and dimensionless variables, and we shall refer to them by this last number.

2. Exact Solutions.

2.1. Time Independent Solution. Let us seek solutions of (1)with $u = u_0 = const.$ and $v = v_0 = const.$ Then, from (1) we obtain:

$$-yv_0 = -\frac{\partial\Phi}{\partial\bar{x}}, \tag{3}$$

$$yu_0 = -\frac{\partial\Phi}{\partial y}, \tag{4}$$

$$u_0\frac{\partial\Phi}{\partial\bar{x}} + v_0\frac{\partial\Phi}{\partial y} = 0. \tag{5}$$

Clearly (3) and (4) imply (5). From (4) we have

$$\Phi(\bar{x}, y) = \Phi_0(\bar{x}, y) = C(x) - \frac{1}{2}u_0y^2. \tag{6}$$

Using (6) in (3),

$$C'(\bar{x}) = yv_0$$

which is possible only if $v_0 = 0, C'(\bar{x}) = 0$, i.e., $C = const.$ We now choose $\bar{H}$ so that $\Phi(\bar{x}, 0) = 1$. We have thus obtained the barotropic solution:

$$u = u_0 = const$$

$$v = v_0 = 0$$

$$\Phi = \Phi_0(y) = 1 - \frac{1}{2}u_0y^2.$$

2.2. Exponentially decaying solution. Let us choose $v \equiv 0$ in the system (1) and look for periodic solutions in $\bar{x}$ and $\bar{t}$ which have the wave-number equal to the frequency. Calling

$$\begin{aligned} x &= k\bar{x}, \\ t &= \omega\bar{t}, \\ z &= x - t, \end{aligned} \tag{8}$$

we then have, with $\omega = k$:

$$-k\frac{\partial u}{\partial z}+ku\frac{\partial u}{\partial z} = -k\frac{\partial \Phi}{\partial z} \tag{9}$$

$$yu = -\frac{\partial \Phi}{\partial y} \tag{10}$$

$$-k\frac{\partial \Phi}{\partial z}+ku\frac{\partial \Phi}{\partial z}+k\Phi\frac{\partial u}{\partial z} = 0 \tag{11}$$

We can treat equations (9) and (11) as a system in $\frac{\partial u}{\partial z}, \frac{\partial \Phi}{\partial z}$, that is:

$$(u-1)\frac{\partial u}{\partial z}+\frac{\partial \Phi}{\partial z} = 0, \tag{12}$$

$$\Phi\frac{\partial u}{\partial z}+(u-1)\frac{\partial \Phi}{\partial z} = 0. \tag{13}$$

The determinant of the system is either zero or the functions u and Φ are independent of z.

In the first case,

$$\Delta = (u-1)^2 - \Phi \equiv 0,$$

then, using (12),

$$3(u-1)\frac{\partial u}{\partial z} = 0.$$

Since we are analyzing the case in which u and Φ are not necessarily independent of z,we must have $u \equiv 1$, then $\Phi \equiv 0$ in contradiction with the statement of equation (10).

In the second case

$$\frac{\partial u}{\partial z} = \frac{\partial \Phi}{\partial z} = 0,$$

and for any decaying function $u(y)$, we obtain from (10) a function $\Phi(y)$,in particular,

$$u = \Phi = e^{-\frac{1}{2}y^2}$$

satisfies the equation, a choice which will be justified later.

3. Non-Linear Problem by PFS Method. Besides the basic solutions of system (1), there may exist, and we shall seek for it, a non-trivial, periodic in x and t, solution which bifurcates from $u_0 = v_0 = 0, \Phi = 1$ Let ω and k be respectively its frequency and wave-number. As before, we utilize the transformations (8).

We shall determine this solution through a development symmilar to the one described in [13]. In order to do that, we set

$$
\begin{aligned}
u &= \sum_{n=1}^{\infty} u_n \varepsilon^n, \\
v &= \sum_{n=1}^{\infty} v_n \varepsilon^n, \\
\Phi &= \sum_{n=0}^{\infty} \Phi_n \varepsilon^n, \\
\omega &= \sum_{n=0}^{\infty} \omega_n \varepsilon^n.
\end{aligned}
$$

Using (3) in (1), collecting and equating powers of ε we obtain:

$$
\begin{aligned}
-\omega_0 \frac{\partial u_n}{\partial z} - y v_n + k\frac{\partial \Phi_n}{\partial z} &= \sum_{j=1}^{n-1}\left\{\omega_j \frac{\partial u_{n-j}}{\partial z} - k u_j \frac{\partial u_{n-j}}{\partial z} - v_j \frac{\partial u_{n-j}}{\partial y}\right\} \\
-\omega_0 \frac{\partial v_n}{\partial z} + y u_n + \frac{\partial \Phi_n}{\partial y} &= \sum_{j=1}^{n-1}\left\{\omega_j \frac{\partial v_{n-j}}{\partial z} - k u_j \frac{\partial v_{n-j}}{\partial z} - v_j \frac{\partial v_{n-j}}{\partial y}\right\} \\
-\omega_0 \frac{\partial \Phi_n}{\partial z} + k\frac{\partial u_n}{\partial z} + \frac{\partial v_n}{\partial y} &= \sum_{j=1}^{n-1}\left\{\omega_j \frac{\partial \Phi_{n-j}}{\partial z} - k u_j \frac{\partial \Phi_{n-j}}{\partial z} - v_j \frac{\partial \Phi_{n-j}}{\partial y}\right. \\
&- \left.\Phi_j\left(k\frac{\partial u_{n-j}}{\partial z} + \frac{\partial v_{n-j}}{\partial y}\right)\right\}.
\end{aligned} \tag{14}
$$

Since we seek solutions which are 2π periodic in x and t,in particular with travelling-wave form, we define

$$
\begin{aligned}
u_n &= \sum_{p=0}^{\infty}\left[u_n^+(y,p)\cos pz + u_n^-(y,p)\sin pz\right], \\
v_n &= \sum_{p=0}^{\infty}\left[v_n^+(y,p)\cos pz + v_n^-(y,p)\sin pz\right], \\
\Phi_n &= \sum_{p=0}^{\infty}\left[\Phi_n^+(y,p)\cos pz + \Phi_n^-(y,p)\sin pz\right],
\end{aligned}
$$

with unknown coefficients to be determined.

3.1. LHS.

3.1.1. First Equation.

$$
-\omega_0 \frac{\partial u_n}{\partial z} - y v_n + k\frac{\partial \Phi_n}{\partial z} = R_1
$$

$$
\begin{aligned}
-\omega_0 \sum_{p=1}^{\infty} p\left[u_n^-(y,p)\cos pz - u_n^+(y,p)\sin pz\right] &- \\
y\sum_{p=0}^{\infty}\left[v_n^+(y,p)\cos pz + v_n^-(y,p)\sin pz\right] &+ \\
k\sum_{p=1}^{\infty} p\left[\Phi_n^-(y,p)\cos pz - \Phi_n^+(y,p)\sin pz\right] &= -y v_n^+(y,0) \\
+ \sum_{p=1}^{\infty}\left[-p\omega_0 u_n^-(y,p) + pk\Phi_n^-(y,p) - y v_n^+(y,p)\right]\cos pz & \\
+ \sum_{p=1}^{\infty}\left[p\omega_0 u_n^+(y,p) - pk\Phi_n^+(y,p) - y v_n^-(y,p)\right]\sin pz. &
\end{aligned}
$$

Then

$$-yv_n^+(y,0) = R_1^+(y,0)$$

and

$$\pm p\omega_0 u_n^\mp(y,p) \mp pk\Phi_n^\pm(y,p) \quad = \quad R_1^\pm(y,p) + yv_n^\pm(y,p).$$

Second Equation.

$$-\omega_0 \frac{\partial v_n}{\partial z} + yu_n + \frac{\partial \Phi_n}{\partial y} = R_2$$

$$\begin{aligned}
-\omega_0 \sum_{p=1}^{\infty} p\left[v_n^-(y,p)\cos pz - v_n^+(y,p)\sin pz\right] \quad &+ \\
y \sum_{p=0}^{\infty} \left[u_n^+(y,p)\cos pz + u_n^-(y,p)\sin pz\right] \quad &+ \\
\sum_{p=0}^{\infty} \left[\frac{d\Phi_n^+(y,p)}{dy}\cos pz + \frac{d\Phi_n^-(y,p)}{dy}\sin pz\right] \quad &= \quad yu_n^+(y,0) + \frac{d\Phi_n^+(y,0)}{dy} \\
- \quad \sum_{p=1}^{\infty} \left[-p\omega_0 v_n^-(y,p) + yu_n^+(y,p) + \frac{d\Phi_n^+(y,p)}{dy}\right]\cos pz & \\
- \quad \sum_{p=1}^{\infty} \left[p\omega_0 v_n^+(y,p) + yu_n^-(y,p) + \frac{d\Phi_n^-(y,p)}{dy}\right]\sin pz. &
\end{aligned}$$

Then

$$yu_n^+(y,0) + \frac{d\Phi_n^+(y,0)}{dy} = R_2^+(y,0)$$

and

$$yu_n^\pm(y,p) + \frac{d\Phi_n^\pm(y,p)}{dy} \mp p\omega_0 v_n^\mp(y,p) \quad = \quad R_2^\pm(y,p).$$

3.1.2. Third Equation.

$$-\omega_0 \frac{\partial \Phi_n}{\partial z} + k\frac{\partial u_n}{\partial z} + \frac{\partial v_n}{\partial y} = R_3$$

$$\begin{aligned}
-\omega_0 \sum_{p=1}^{\infty} p\left[\Phi_n^-(y,p)\cos pz - \Phi_n^+(y,p)\sin pz\right] \quad &+ \\
k \sum_{p=1}^{\infty} p\left[u_n^-(y,p)\cos pz - u_n^+(y,p)\sin pz\right] \quad &+ \\
\sum_{p=0}^{\infty} \left[\frac{dv_n^+(y,p)}{dy}\cos pz + \frac{dv_n^-(y,p)}{dy}\sin pz\right] \quad &= \quad \frac{dv_n^+(y,0)}{dy} \\
+ \quad \sum_{p=1}^{\infty} \left[pku_n^-(y,p) - p\omega_0\Phi_n^-(y,p) + \frac{dv_n^+(y,p)}{dy}\right]\cos pz & \\
+ \quad \sum_{p=1}^{\infty} \left[-pku_n^+(y,p) + p\omega_0\Phi_n^+(y,p) + \frac{dv_n^-(y,p)}{dy}\right]\sin pz. &
\end{aligned}$$

Then

$$\frac{dv_n^+(y,0)}{dy} = R_3^+(y,0)$$

and

$$\mp pku_n^\pm(y,p) \pm p\omega_0\Phi_n^\pm(y,p) = R_3^\mp(y,p) - \frac{dv_n^\mp(y,p)}{dy}.$$

3.1.3. Summary of Equations.

$$\begin{aligned} -yv_n^+(y,0) &= R_1^+(y,0), \\ yu_n^+(y,0) + \frac{d\Phi_n^+(y,0)}{dy} &= R_2^+(y,0), \\ \frac{dv_n^+(y,0)}{dy} &= R_3^+(y,0). \end{aligned}$$

$$\pm p\omega_0 u_n^\pm(y,p) \mp pk\Phi_n^\pm(y,p) - yv_n^\mp(y,p) = R_1^\pm(y,p), \tag{15}$$

$$yu_n^\pm(y,p) + \frac{d\Phi_n^\pm(y,p)}{dy} \mp p\omega_0 v_n^\mp(y,p) = R_2^\pm(y,p), \tag{16}$$

$$\mp pku_n^\pm(y,p) \pm p\omega_0\Phi_n^\pm(y,p) + \frac{dv_n^\mp(y,p)}{dy} = R_3^\pm(y,p), \tag{17}$$

3.2. Case $n = 1$. In this case $R_1^\mp(y,p) = R_2^\pm(y,p) = R_3^\mp(y,p) = 0$. We now set $n = 1$ and treat equations (15) and (17) as a system in the unkwnowns $u_n^\pm(y,p)$, $\Phi_n^\pm(y,p)$. The determinant of this systems is

$$\Delta_1 = \omega_0^2 - k^2$$

3.3. Subcase $\omega_0 = \pm k$. If $\omega_0 = k$, adding the equations (15) and (17) we get

$$\frac{dv_1^\mp(y,1)}{dy} = yv_1^\pm(y,1).$$

Therefore

$$v_1^\pm(y,1) = 0,$$

otherwise $v_1^\pm(y,1)$ would be exponentially increasing, and from (15):

$$u_1^+(y,1) = \Phi_1^+(y,1).$$

Using this in (16):

$$\pm\omega_0 v_1^\pm(y,1) + yu_1^\mp(y,1) + \frac{d\Phi_1^\mp(y,1)}{dy} = y\Phi_1^\mp + \frac{d\Phi_1^\mp(y,p)}{dy} = 0.$$

Then

$$\Phi_1^{\pm}(y,1) = u_1^{\pm}(y,1) = A^{\pm} e^{\frac{1}{2}y^2}.$$

We have thus recovered the exponentially decaying solution found in page 3 and, furthermore, justified the choice done there. If $\omega_0 = -k$ subtracting (17) from (15) we get

$$\frac{dv_1^{\pm}(y,1)}{dy} = -yv_1^{\pm}(y,1),$$

hence

$$v_1^{\pm}(y,1) = A^{\pm} e^{-\frac{1}{2}y^2}.$$

And from (15),(16)

$$\begin{aligned} \mp k u_1^{\pm}(y,1) \mp k\Phi_1^{\pm}(y,1) &= A^{\pm} y e^{\frac{1}{2}y^2} \\ y u_1^{\pm}(y,1) + \frac{d\Phi_1^{\pm}(y,1)}{dy} &= \pm k A^{\pm} e^{\frac{1}{2}y^2} \end{aligned}$$

Multiplying the first eequation by y, the second one by $\pm k$ and adding we have

$$\pm k \frac{d\Phi_1^{\pm}(y,1)}{dy} \mp k y \Phi_1^{\pm}(y,1) = \left(k^2 + y^2\right) A^{\pm} e^{-\frac{1}{2}y^2}$$

:

$$\frac{d\Phi_1^{\pm}(y,1)}{dy} - y\Phi_1^{\pm}(y,1) = \pm\frac{A^{\pm}}{k}\left(k^2 + y^2\right) e^{-\frac{1}{2}y^2}.$$

Recalling that

$$\int e^{-y^2} dy = const + \frac{\sqrt{\pi}}{2}\,\mathrm{erf}(y)$$

and

$$\int y^2 e^{-y^2} dy = -\frac{1}{2} y e^{-y^2} + \frac{1}{2}\int e^{-y^2} dy$$

we have

$$\frac{d}{dy}\left[e^{-\frac{1}{2}y^2}\Phi_1^{\pm}(y,1)\right] = \pm\frac{A^{\pm}}{k}\left(k^2 + y^2\right) e^{-y^2}$$

therefore

$$
\begin{aligned}
e^{-\frac{1}{2}y^2}\Phi_1^{\pm}(y,1) &= \pm \frac{A^{\pm}}{k}\left\{k^2\left[c_1+\sqrt{\pi}\,\mathrm{erf}(y)\right]+\frac{c_2}{2}+\frac{\sqrt{\pi}}{2}\,\mathrm{erf}(y)-\frac{1}{2}ye^{-y^2}\right\} \\
&= \pm \frac{A^{\pm}}{k}\left\{k^2c_1+\frac{c_2}{2}+(k^2+1)\frac{\sqrt{\pi}}{2}\,\mathrm{erf}(y)-\frac{1}{2}ye^{-y^2}\right\} \\
&= \pm \frac{A^{\pm}}{k}\left\{c+(k^2+1)\frac{\sqrt{\pi}}{2}\,\mathrm{erf}(y)-\frac{1}{2}ye^{-y^2}\right\}
\end{aligned}
$$

Then

$$
\Phi_1^{\pm}(y,1) = \mp\frac{A^{\pm}}{2k}ye^{-\frac{1}{2}y^2} \pm \frac{A^{\pm}}{k}\left[c+\left(k^2+1\right)\frac{\sqrt{\pi}}{2}\,\mathrm{erf}(y)\right]e^{\frac{1}{2}y^2}.
$$

Since (see [5] pp 298)

$$
\frac{1}{y+\sqrt{y^2+2}} < e^{y^2}\int_y^{\infty} e^{-y^2}\,dy < \frac{1}{y+\sqrt{y^2+\frac{4}{\pi}}}
$$

we have that

$$
\frac{\sqrt{\pi}e^{-\frac{y^2}{2}}}{2\left(y+\sqrt{y^2+2}\right)} < \frac{\sqrt{\pi}}{2}e^{\frac{y^2}{2}}\,\mathrm{erfc}(y) < \frac{\sqrt{\pi}e^{-\frac{y^2}{2}}}{2\left(y+\sqrt{y^2+\frac{4}{\pi}}\right)}.
$$

Therefore

$$
\Phi_1^{\pm}(y,1) = \mp\frac{A^{\pm}}{2k}ye^{-\frac{1}{2}y^2} \pm \frac{A^{\pm}}{k}\left[c+\left(k^2+1\right)\frac{\sqrt{\pi}}{2}\right]e^{\frac{1}{2}y^2} \mp \frac{A^{\pm}}{k}\left(k^2+1\right)\frac{\sqrt{\pi}}{2}\,\mathrm{erfc}(y)e^{\frac{1}{2}y^2}.
$$

Since we expect the function to decay exponentially we must have

$$
c = -\left(k^2+1\right)\frac{\sqrt{\pi}}{2}.
$$

With this

$$
\Phi_1^{\pm}(y,1) = \mp\frac{A^{\pm}}{2k}ye^{-\frac{1}{2}y^2} \mp \frac{A^{\pm}}{k}\left(k^2+1\right)\frac{\sqrt{\pi}}{2}e^{\frac{1}{2}y^2}\,\mathrm{erfc}(y)
$$

and

$$
\begin{aligned}
ku_1^{\pm}(y,1) &= \mp A^{\pm}ye^{-\frac{1}{2}y^2} - k\Phi_1^{\pm}(y,1) \\
&= \mp A^{\pm}ye^{-\frac{1}{2}y^2} \pm \frac{A^{\pm}}{2}ye^{-\frac{1}{2}y^2} \pm A^{\pm}(k^2+1)\sqrt{\pi}e^{\frac{1}{2}y^2}\,\mathrm{erfc}(y) \\
&= \mp \frac{A^{\pm}}{2}ye^{-\frac{1}{2}y^2} \pm A^{\pm}(k^2+1)\sqrt{\pi}e^{\frac{1}{2}y^2}\,\mathrm{erfc}(y).
\end{aligned}
$$

$$u_1^{\pm}(y,1) = \mp\frac{A^{\pm}}{2k} y e^{-\frac{1}{2}y^2} \pm \frac{A^{\pm}}{k}\left(k^2+1\right)\sqrt{\pi} e^{\frac{1}{2}y^2}\,\mathrm{erfc}(y).$$

3.4. Subcase $\omega_0 \neq \pm k$. In this case $\Delta_1 \neq 0$, therefore we can solve the system (15.17) in terms of $u_1^{\pm}(y,1)$, $\Phi_1^{\pm}(y,1)$, obtaining

$$\begin{aligned} u_1^{\pm}(y,1) &= \pm\frac{1}{\Delta_1}\left[\omega_0 y v_1^{\mp}(y,1) - k\frac{dv_1^{\mp}(y,1)}{dy}\right], \\ \Phi_1^{\pm}(y,1) &= \pm\frac{1}{\Delta_1}\left[k y v_1^{\mp}(y,1) - \omega_0\frac{dv_1^{\mp}(y,1)}{dy}\right]. \end{aligned}$$

Using these results in (16)

$$\begin{aligned} & y u_1^{\mp}(y,1) + \frac{d\Phi_1^{\mp}(y,1)}{dy} \mp \omega_0 v_1^{\mp}(y,1) = \\ & = \frac{y}{\Delta_1}\left[\omega_0 y v_1^{\mp}(y,1) - k\frac{dv_1^{\mp}(y,1)}{dy}\right] \pm \frac{1}{\Delta_1}\frac{d}{dy}\left[k y v_1^{\mp}(y,1) - \omega_0\frac{dv_1^{\mp}(y,1)}{dy}\right] \mp \omega_0 v_1^{\mp}(y,1) = \\ & = \left\{\frac{\omega_0 y^2}{\Delta_1} + \frac{k}{\Delta_1} - \omega_0\right\} v_1^{\mp}(y,1) + \left\{\mp\frac{yk}{\Delta_1} \pm \frac{ky}{\Delta_1}\right\}\frac{dv_1^{\mp}(y,1)}{dy} \mp \frac{\omega_0}{\Delta_1}\frac{d^2 v_1^{\mp}(y,1)}{dy^2}. \\ & = \frac{\omega_0}{\Delta_1}\frac{d^2 v_1^{\mp}(y,1)}{dy^2} \pm \left\{\frac{\omega_0 y^2}{\Delta_1} + \frac{k}{\Delta_1} - \omega_0\right\} v_1^{\mp}(y,1) = 0. \end{aligned}$$

That is

(18)
$$\frac{d^2 v_1^{\mp}(y,1)}{dy^2} + \left\{\omega_0^2 - \frac{k}{\omega_0} - k^2 - y^2\right\} v_1^{\mp}(y,1) = 0.$$

a Schrödinger Equation[4] (E. Butkov, pp 472 – 474), whose solution is

(19)
$$v_1^{\pm}(y,1) = A^{\pm} e^{-y^2/2} H_m(y), \quad m \geq 0,$$

where $H_m(y)$ is the Hermite polynomial of degree m, provided

(20)
$$\omega_0^2 - \frac{k}{\omega_0} - k^2 = 2m+1 \quad m \geq 0.$$

The dispersion equation (20) yields:

$$\begin{aligned} k &= \frac{1}{2}\left(-\frac{1}{\omega_0} \pm \sqrt{\left(\frac{1}{\omega_0}\right)^2 - 4\left(2m+1-\omega_0^2\right)}\right) \\ &= \frac{1}{2}\left\{-\frac{1}{\omega_0} \pm \frac{1}{|\omega_0|}\sqrt{1-4\left(2m+1-\omega_0^2\right)\omega_0^2}\right\} \\ &= -\frac{1}{2\omega_0}\left(1 \mp \mathrm{sgn}\left(\omega_0\right)\sqrt{1-4\left(2m+1-\omega_0^2\right)\omega_0^2}\right) \\ &= -\frac{1}{2\omega_0}\left(1 \mp \mathrm{sgn}\left(\omega_0\right)\sqrt{1-8m\omega_0^2-4\omega_0^2+4\omega_0^4}\right) \\ &= -\frac{1}{2\omega_0}\left(1 \mp \mathrm{sgn}\left(\omega_0\right)\sqrt{\left(1-2\omega_0^2\right)^2-8m\omega_0^2}\right). \end{aligned}$$

One must observe that when $m = 0$,

$$k = -\frac{1}{2\omega_0}\left[1 \mp \operatorname{sgn}(\omega_0)\sqrt{(1-2\omega_0^2)^2}\right] = -\frac{1}{2\omega_0}\left[1 \mp \operatorname{sgn}(\omega_0)\left|1-2\omega_0^2\right|\right],$$

being the only case in which for any given frequency ω there exist three values of the wave-number k, because for $m \neq 0$, the discriminant is

$$4\omega_0^4 - 4(2m+1)\omega_0^2 + 1$$

which is negative in the set

$$\left(-a^+, -a^-\right) \cup \left(a^-, a^+\right),$$

where

$$a^\pm = \sqrt{\frac{2m+1 \pm \sqrt{(2m+1)^2 - 1}}{2}}.$$

Care must also be taken in this case with the definition of the different solution branches. For $0 < \omega_0 < 1/\sqrt{2}$ we have

$$k = -\frac{1}{2\omega_0}\left[1 \mp \left(1 - 2\omega_0^2\right)\right] = \begin{cases} -\omega_0 \\ \omega_0 - \dfrac{1}{\omega_0} \end{cases},$$

while for $1/\sqrt{2} < \omega_0$

$$k = -\frac{1}{2\omega_0}\left[1 \pm \left(1 - 2\omega_0^2\right)\right] = \begin{cases} \omega_0 - \dfrac{1}{\omega_0} \\ -\omega_0 \end{cases}.$$

Thus, graphically, one can see in Figure 1 how the branches continue.

Finally, we have:

$$\begin{aligned}
u_1^\pm(y,1) &= \pm\frac{1}{\Delta_1}\left\{y\omega_0 v_1^\mp(y,1) - k\frac{dv_1^\mp(y,1)}{dy}\right\} \\
&= \pm\frac{A^\mp}{\Delta_1}\left\{y\omega_0 e^{-y^2/2}H_m(y) - k\frac{de^{-y^2/2}H_m(y)}{dy}\right\} \\
&= \pm\frac{A^\mp}{\Delta_1}\left\{y\omega_0 e^{-y^2/2}H_m(y) + kye^{-y^2/2}H_m(y) - ke^{-y^2/2}H_m'(y)\right\} \\
&= \pm\frac{A^\mp}{\Delta_1}e^{-y^2/2}\left\{(\omega_0 + k)\, yH_m(y) - 2mkH_{m-1}(y)\right\} \\
&= \pm\frac{A^\mp}{2\Delta_1}e^{-y^2/2}\left\{(\omega_0 + k)\, H_{m+1}(y) + 2m\,(\omega_0 - k)\, H_{m-1}(y)\right\}
\end{aligned}$$

FIG. 1. *Relationship between ω and k obtained from the dispersion equation (3.8) with $m = 0$.*

FIG. 2. *Relationship between ω and k obtained from the dispersion equation with $n = 1$.*

$$
\begin{aligned}
\Phi_1^{\pm}(y,1) &= \pm\frac{1}{\Delta_1}\left\{ykv_1^{\mp}(y,1) - \omega_0\frac{dv_1^{\mp}(y,1)}{dy}\right\} \\
&= \pm\frac{A^{\mp}}{\Delta_1}\left\{yke^{-y^2/2}H_m(y) - \omega_0\frac{de^{-y^2/2}H_m(y)}{dy}\right\} \\
&= \pm\frac{A^{\mp}}{\Delta_1}e^{-y^2/2}\left\{yke^{-y^2/2}H_m(y) + \omega_0 ye^{-y^2/2}H_m(y) - \omega_0 e^{-y^2/2}H_m'(y)\right\} \\
&= \pm\frac{A^{\mp}}{\Delta_1}e^{-y^2/2}\left\{(\omega_0 + k)\,yH_m(y) - 2m\omega_0 H_{m-1}(y)\right\} \\
&= \pm\frac{A^{\mp}}{2\Delta_1}e^{-y^2/2}\left\{(\omega_0 + k)\,H_{m+1}(y) - 2m\,(\omega_0 - k)\,H_{m-1}(y)\right\}.
\end{aligned}
$$

3.4.1. Summary.

$$
\begin{aligned}
u_1^{\pm}(y,1) &= \pm\frac{A^{\mp}}{2\Delta_1}e^{-y^2/2}\left\{(\omega_0+k)\,H_{m+1}(y)+2m\,(\omega_0-k)\,H_{m-1}(y)\right\},\\
v_1^{\pm}(y,1) &= A^{\pm}e^{-y^2/2}H_m(y),\\
\Phi_1^{\pm}(y,1) &= \pm\frac{A^{\mp}}{2\Delta_1}e^{-y^2/2}\left\{(\omega_0+k)\,H_{m+1}(y)-2m\,(\omega_0-k)\,H_{m-1}(y)\right\}.
\end{aligned}
$$

We have thus completely solved the problem for $n=1$. It is important to observe that we have introduced the unknown frequency in the equation and this fixes it when the dispersion equation is solved. However, had we only introduced the wave-number, then for any given k, we obtain three values of ω_0 and, therefore, the kernel of the variational equations has dimension six. In fact, the solution, depending on the initial conditions, may consist in the co-existence of the three different types of waves. Furthermore, this implies that the problem not necessarily has a solution for $n>1$ and $p=1$, because the right-hand-sides of the equations should be orthogonal to the kernel of the adjoint operator defined by the variational equations. This amounts to six conditions and therefore, it would be necessary to have additional parameters (a total of six).

3.5. Case $n>1$. For values of $n>1$, it follows from (14) that the systems with right-hand-sides given by (15-17) are no longer homogeneous.

Solving the system formed by the equations (15) and (17) we obtain:

$$
\begin{aligned}
u_n^{\pm}(y,p) &= \pm\frac{\omega_0}{p\Delta_1}\left[R_1^{\mp}(y,p)+yv_n^{\mp}(y,p)\right]\pm\frac{k}{p\Delta_1}\left[R_3^{\mp}(y,p)-\frac{dv_n^{\mp}(y,p)}{dy}\right]\\
&= \pm\frac{1}{p\Delta_1}\left\{\left[\omega_0R_1^{\mp}(y,p)+kR_3^{\mp}(y,p)\right]+\left[\omega_0yv_n^{\mp}(y,p)-k\frac{dv_n^{\mp}(y,p)}{dy}\right]\right\}.
\end{aligned}
$$

$$
\begin{aligned}
\Phi_n^{\pm}(y,p) &= \pm\frac{k}{p\Delta_1}\left[R_1^{\mp}(y,p)+yv_n^{\mp}(y,p)\right]\pm\frac{\omega_0}{p\Delta_1}\left[R_3^{\mp}(y,p)-\frac{dv_n^{\mp}(y,p)}{dy}\right]\\
&= \pm\frac{1}{p\Delta_1}\left\{\left[kR_1^{\mp}(y,p)+\omega_0R_3^{\mp}(y,p)\right]+\left[kyv_n^{\mp}(y,p)-\omega_0\frac{dv_n^{\mp}(y,p)}{dy}\right]\right\}.
\end{aligned}
$$

Using these results in (16)

$$
\begin{aligned}
&yu_n^{\pm}(y,p)+\frac{d\Phi_n^{\pm}(y,p)}{dy}\mp p\omega_0v_n^{\mp}(y,p)=\\
&=\pm\frac{y}{p\Delta_1}\left\{\left[\omega_0R_1^{\mp}(y,p)+kR_3^{\mp}(y,p)\right]+\left[\omega_0yv_n^{\mp}(y,p)-k\frac{dv_n^{\mp}(y,p)}{dy}\right]\right\}\\
&\pm\frac{1}{p\Delta_1}\frac{d}{dy}\left\{\left[kR_1^{\mp}(y,p)+\omega_0R_3^{\mp}(y,p)\right]+\left[kyv_n^{\mp}(y,p)-\omega_0\frac{dv_n^{\mp}(y,p)}{dy}\right]\right\}\mp p\omega_0v_n^{\mp}(y,p)=\\
&=\pm\frac{1}{p\Delta_1}\left\{y\left[\omega_0R_1^{\mp}(y,p)+kR_3^{\mp}(y,p)\right]+\frac{d}{dy}\left[kR_1^{\mp}(y,p)+\omega_0R_3^{\mp}(y,p)\right]\right\}\\
&\pm\frac{1}{p\Delta_1}\left\{y\left[\omega_0yv_n^{\mp}(y,p)-k\frac{dv_n^{\mp}(y,p)}{dy}\right]+\frac{d}{dy}\left[kyv_n^{\mp}(y,p)-\omega_0\frac{dv_n^{\mp}(y,p)}{dy}\right]\right\}\mp p\omega_0v_n^{\mp}(y,p)=
\end{aligned}
$$

$$\pm\frac{1}{p\Delta_1}\left\{y\left[\omega_0 R_1^{\pm}(y,p)+kR_3^{\pm}(y,p)\right]+\frac{d}{dy}\left[kR_1^{\pm}(y,p)+\omega_0 R_3^{\pm}(y,p)\right]\right\}$$
$$\pm\left\{\frac{\omega_0 y^2}{p\Delta_1}+\frac{k}{p\Delta_1}-p\omega_0\right\}v_n^{\pm}(y,p)+\left\{\mp\frac{yk}{p\Delta_1}\pm\frac{ky}{p\Delta_1}\right\}\frac{dv_n^{\pm}(y,p)}{dy}\mp\frac{\omega_0}{p\Delta_1}\frac{d^2v_n^{\pm}(y,p)}{dy^2}.$$

Therefore

$$\mp\frac{\omega_0}{p\Delta_1}\frac{d^2v_n^{\pm}(y,p)}{dy^2}\pm\left\{\frac{\omega_0 y^2}{p\Delta_1}+\frac{k}{p\Delta_1}-p\omega_0\right\}v_n^{\pm}(y,p)$$
$$\pm\frac{1}{p\Delta_1}\left\{y\left[\omega_0 R_1^{\mp}(y,p)+kR_3^{\pm}(y,p)\right]+\frac{d}{dy}\left[kR_1^{\mp}(y,p)+\omega_0 R_3^{\pm}(y,p)\right]\right\}=R_2^{\pm}$$

or

(21)
$$\begin{aligned}\frac{d^2v_n^{\pm}(y,p)}{dy^2}+\left\{p^2\Delta_1-\frac{k}{\omega_0}-y^2\right\}v_n^{\pm}(y,p) &= \mp\frac{p\Delta_1}{\omega_0}R_2^{\pm}\\ +\frac{1}{\omega_0}\left\{y\left[\omega_0 R_1^{\mp}(y,p)+kR_3^{\pm}(y,p)\right]\right. &+ \left.\frac{d}{dy}\left[kR_1^{\pm}(y,p)+\omega_0 R_3^{\mp}(y,p)\right]\right\}.\end{aligned}$$

When $p=1$, the left-hand-side of equation (21) is equal to the left-hand-side of equation (18) and, therefore, we already know that we have a singular operator, hence, the right-hand-side must be orthogonal to the kernel of the adjoint operator. One may also have a singular operator for other values of p. To see this we can proceed as it follows:

$$p^2\Delta_1-\frac{k}{\omega_0}=p^2\Delta_1-\Delta_1+\Delta_1-\frac{k}{\omega_0}=(p^2-1)\Delta_1+2m+1$$

This expression is equal to $2m'+1$ for some m' if

(22)
$$\omega_0^2-k^2=\frac{2(m'-m)}{p^2-1}$$

Thus, chosen any m, m' and $p\neq 1$, (22) represents a hyperbola, and if they are adequately chosen, the hyperbola will intersect the curves defined by the dispersion equation. This means that for a possibly infinite, but discrete, set of values of ω_0 and k, there is coalescence of solutions with different Hermite polynomials.

Assuming that ω_0 and k are not related in such a way, then it is still necessary to determine the conditions under which the problem may have a solution, namely, that the coefficient of ω_1 be different from zero.

3.6. Solvability Condition. In order to determine the solvability condition necessary for the solution of the problem, we must analyze the second-order approximation and evaluate the coefficient of ω_1. Thus we have:

$$\begin{aligned}u_1\frac{\partial u_1}{\partial z} &= \left[u_1^{+}\cos z+u_1^{-}\sin z\right]\left[u_1^{-}\cos z-u_1^{+}\sin z\right]\\ &= \frac{1}{2}u_1^{+}u_1^{-}(1+\cos 2z)-\frac{1}{2}u_1^{-}u_1^{+}(1-\cos 2z)+\frac{1}{2}\left[\left(u_1^{-}\right)^2-\left(u_1^{+}\right)^2\right]\sin 2z\\ &= \frac{1}{2}u_1^{+}u_1^{-}-\frac{1}{2}u_1^{-}u_1^{+}+\left\{\frac{1}{2}u_1^{+}u_1^{-}+\frac{1}{2}u_1^{-}u_1^{+}\right\}\cos 2z+\frac{1}{2}\left[\left(u_1^{-}\right)^2-\left(u_1^{+}\right)^2\right]\sin 2z\\ &= u_1^{+}u_1^{-}\cos 2z+\frac{1}{2}\left\{\left(u_1^{-}\right)^2-\left(u_1^{+}\right)^2\right\}\sin 2z.\end{aligned}$$

$$
\begin{aligned}
v_1 \frac{\partial u_1}{\partial y} &= \left[v_1^+ \cos z + v_1^- \sin z\right] \left[\frac{\partial u_1^+}{\partial y} \cos z + \frac{\partial u_1^-}{\partial y} \sin z\right] \\
&= \frac{1}{2} v_1^+ \frac{\partial u_1^+}{\partial y} (1 + \cos 2z) + \frac{1}{2} v_1^- \frac{\partial u_1^-}{\partial y} (1 - \cos 2z) + \frac{1}{2}\left[v_1^+ \frac{\partial u_1^-}{\partial y} + v_1^- \frac{\partial u_1^+}{\partial y}\right] \sin 2z \\
&= \frac{1}{2}\left\{v_1^+ \frac{\partial u_1^+}{\partial y} + v_1^- \frac{\partial u_1^-}{\partial y}\right\} + \frac{1}{2}\left\{v_1^+ \frac{\partial u_1^+}{\partial y} - v_1^- \frac{\partial u_1^-}{\partial y}\right\} \cos 2z \\
&+ \frac{1}{2}\left\{v_1^+ \frac{\partial u_1^-}{\partial y} + v_1^- \frac{\partial u_1^+}{\partial y}\right\} \sin 2z.
\end{aligned}
$$

$$
\begin{aligned}
&R_1 = \omega_1 \frac{\partial u_1}{\partial z} - k u_1 \frac{\partial u_1}{\partial z} - v_1 \frac{\partial u_1}{\partial y} = \\
&\omega_1 \left[u_1^- \cos z - u_1^+ \sin z\right] \\
&-k\left\{u_1^+ u_1^-\right\} \cos 2z - \frac{k}{2}\left\{\left(u_1^-\right)^2 - \left(u_1^+\right)^2\right\} \sin 2z \\
&-\frac{1}{2}\left\{v_1^+ \frac{\partial u_1^+}{\partial y} + v_1^- \frac{\partial u_1^-}{\partial y}\right\} - \frac{1}{2}\left\{v_1^+ \frac{\partial u_1^+}{\partial y} - v_1^- \frac{\partial u_1^-}{\partial y}\right\} \cos 2z - \frac{1}{2}\left\{v_1^+ \frac{\partial u_1^-}{\partial y} + v_1^- \frac{\partial u_1^+}{\partial y}\right\} \sin 2z
\end{aligned}
$$

$$
\begin{aligned}
R_1 &= \omega_1 \left[u_1^- \cos z - u_1^+ \sin z\right] - \frac{1}{2}\left\{v_1^+ \frac{\partial u_1^+}{\partial y} + v_1^- \frac{\partial u_1^-}{\partial y}\right\} \\
&- \frac{1}{2}\left\{2k u_1^+ u_1^- + v_1^+ \frac{\partial u_1^+}{\partial y} - v_1^- \frac{\partial u_1^-}{\partial y}\right\} \cos 2z \\
&- \frac{1}{2}\left\{k\left[\left(u_1^-\right)^2 - \left(u_1^+\right)^2\right] + v_1^+ \frac{\partial u_1^-}{\partial y} + v_1^- \frac{\partial u_1^-}{\partial y}\right\} \sin 2z.
\end{aligned}
$$

$$
\begin{aligned}
\frac{[illegible]}{\partial z} &= \left[u_1^+ \cos z + u_1^- \sin z\right] \left[v_1^- \cos z - v_1^+ \sin z\right] \\
&= \frac{1}{2} u_1^+ v_1^- (1 + \cos 2z) - \frac{1}{2} u_1^- v_1^+ (1 - \cos 2z) + \frac{1}{2}\left(u_1^- v_1^- - u_1^+ v_1^+\right) \sin 2z \\
&= \frac{1}{2}\left\{u_1^+ v_1^- - u_1^- v_1^+\right\} + \frac{1}{2}\left\{u_1^+ v_1^- + u_1^- v_1^+\right\} \cos 2z + \frac{1}{2}\left\{u_1^- v_1^- - u_1^+ v_1^+\right\} \sin 2z
\end{aligned}
$$

$$
\begin{aligned}
v_1 \frac{\partial v_1}{\partial y} &= \left[v_1^+ \cos z + v_1^- \sin z\right] \left[\frac{\partial v_1^+}{\partial y} \cos z + \frac{\partial v_1^-}{\partial y} \sin z\right] \\
&= \frac{1}{2} v_1^+ \frac{\partial v_1^+}{\partial y} (1 + \cos 2z) + \frac{1}{2} v_1^- \frac{\partial v_1^-}{\partial y} (1 - \cos 2z) + \frac{1}{2}\left(v_1^+ \frac{\partial v_1^-}{\partial y} + v_1^- \frac{\partial v_1^+}{\partial y}\right) \sin 2z \\
&= \frac{1}{2}\left\{v_1^+ \frac{\partial v_1^+}{\partial y} + v_1^- \frac{\partial v_1^-}{\partial y}\right\} + \frac{1}{2}\left\{v_1^+ \frac{\partial v_1^+}{\partial y} - v_1^- \frac{\partial v_1^-}{\partial y}\right\} \cos 2z \\
&+ \frac{1}{2}\left\{v_1^+ \frac{\partial v_1^-}{\partial y} + v_1^- \frac{\partial v_1^+}{\partial y}\right\} \sin 2z
\end{aligned}
$$

$$
\begin{aligned}
&R_2 = \omega_1 \frac{\partial v_1}{\partial z} - k u_1 \frac{\partial v_1}{\partial z} - v_1 \frac{\partial v_1}{\partial y} = \\
&\omega_1 \left[v_1^- \cos z - v_1^+ \sin z\right]
\end{aligned}
$$

$$-\frac{k}{2}\left\{u_1^+ v_1^- - u_1^- v_1^+\right\} - \frac{k}{2}\left\{u_1^+ v_1^- + u_1^- v_1^+\right\}\cos 2z - \frac{k}{2}\left\{u_1^- v_1^- - u_1^+ v_1^+\right\}\sin 2z$$
$$-\frac{1}{2}\left\{v_1^+ \frac{\partial v_1^+}{\partial y} + v_1^- \frac{\partial v_1^-}{\partial y}\right\} - \frac{1}{2}\left\{v_1^+ \frac{\partial v_1^+}{\partial y} - v_1^- \frac{\partial v_1^-}{\partial y}\right\}\cos 2z - \frac{1}{2}\left\{v_1^+ \frac{\partial v_1^-}{\partial y} + v_1^- \frac{\partial v_1^+}{\partial y}\right\}\sin 2z$$

$$\begin{aligned}
R_2 &= \omega_1\left[v_1^- \cos z - v_1^+ \sin z\right] - \frac{1}{2}\left\{k\left(u_1^+ v_1^- - u_1^- v_1^+\right) + v_1^+ \frac{\partial v_1^+}{\partial y} + v_1^- \frac{\partial v_1^-}{\partial y}\right\} \\
&- \frac{1}{2}\left\{k\left(u_1^+ v_1^- + u_1^- v_1^+\right) + v_1^+ \frac{\partial v_1^+}{\partial y} - v_1^- \frac{\partial v_1^-}{\partial y}\right\}\cos 2z \\
&- \frac{1}{2}\left\{k\left(u_1^- v_1^- - u_1^+ v_1^+\right) + v_1^+ \frac{\partial v_1^-}{\partial y} + v_1^- \frac{\partial v_1^+}{\partial y}\right\}\sin 2z.
\end{aligned}$$

$$\begin{aligned}
u_1\frac{\partial \Phi_1}{\partial z} &= \left[u_1^+ \cos z + u_1^- \sin z\right]\left[\Phi_1^- \cos z - \Phi_1^+ \sin z\right] \\
&= \frac{1}{2}u_1^+ \Phi_1^- (1+\cos 2z) - \frac{1}{2}u_1^- \Phi_1^+ (1-\cos 2z) + \frac{1}{2}\left(u_1^- \Phi_1^- - u_1^+ \Phi_1^+\right)\sin 2z \\
&= \frac{1}{2}\left\{u_1^+ \Phi_1^- - u_1^- \Phi_1^+\right\} + \frac{1}{2}\left\{u_1^+ \Phi_1^- + u_1^- \Phi_1^+\right\}\cos 2z + \frac{1}{2}\left\{u_1^- \Phi_1^- - u_1^+ \Phi_1^+\right\}\sin 2z
\end{aligned}$$

$$\begin{aligned}
v_1\frac{\partial \Phi_1}{\partial y} &= \left[v_1^+ \cos z + v_1^- \sin z\right]\left[\frac{\partial \Phi_1^+}{\partial y}\cos z + \frac{\partial \Phi_1^-}{\partial y}\sin z\right] \\
&= \frac{1}{2}v_1^+ \frac{\partial \Phi_1^+}{\partial y}(1+\cos 2z) + \frac{1}{2}v_1^- \frac{\partial \Phi_1^-}{\partial y}(1-\cos 2z) \\
&+ \frac{1}{2}\left(v_1^+ \frac{\partial \Phi_1^-}{\partial y} + v_1^- \frac{\partial \Phi_1^+}{\partial y}\right)\sin 2z \\
&= \frac{1}{2}\left\{v_1^+ \frac{\partial \Phi_1^+}{\partial y} + v_1^- \frac{\partial \Phi_1^-}{\partial y}\right\} + \frac{1}{2}\left\{v_1^+ \frac{\partial \Phi_1^+}{\partial y} - v_1^- \frac{\partial \Phi_1^-}{\partial y}\right\}\cos 2z \\
&+ \frac{1}{2}\left\{v_1^+ \frac{\partial \Phi_1^-}{\partial y} + v_1^- \frac{\partial \Phi_1^+}{\partial y}\right\}\sin 2z
\end{aligned}$$

$$\begin{aligned}
k\frac{\partial u_1}{\partial z} + \frac{\partial v_1}{\partial y} &= k\left[u_1^- \cos z - u_1^+ \sin z\right] + \frac{\partial v_1^+}{\partial y}\cos z + \frac{\partial v_1^-}{\partial y}\sin z \\
&= \left(ku_1^- + \frac{\partial v_1^+}{\partial y}\right)\cos z - \left(ku_1^+ - \frac{\partial v_1^-}{\partial y}\right)\sin z
\end{aligned}$$

$$
\begin{aligned}
\Phi_1(k\frac{\partial u_1}{\partial z}+\frac{\partial v_1}{\partial y}) &= \left[\Phi_1^{+}\cos z+\Phi_1^{-}\sin z\right]\left[\left(ku_1^{-}+\frac{\partial v_1^{+}}{\partial y}\right)\cos z-\left(ku_1^{+}-\frac{\partial v_1^{-}}{\partial y}\right)\sin z\right]\\
&= \frac{1}{2}\Phi_1^{+}\left(ku_1^{-}+\frac{\partial v_1^{+}}{\partial y}\right)(1+\cos 2z)-\frac{1}{2}\Phi_1^{-}\left(ku_1^{+}-\frac{\partial v_1^{-}}{\partial y}\right)(1-\cos 2z)\\
&+ \frac{1}{2}\left[\Phi_1^{-}\left(ku_1^{-}+\frac{\partial v_1^{+}}{\partial y}\right)-\Phi_1^{+}\left(ku_1^{+}-\frac{\partial v_1^{-}}{\partial y}\right)\right]\sin 2z\\
&= \frac{1}{2}\left\{\Phi_1^{+}\left(ku_1^{-}+\frac{\partial v_1^{+}}{\partial y}\right)-\Phi_1^{-}\left(ku_1^{+}-\frac{\partial v_1^{-}}{\partial y}\right)\right\}\\
&+ \frac{1}{2}\left\{\Phi_1^{+}\left(ku_1^{-}+\frac{\partial v_1^{+}}{\partial y}\right)+\Phi_1^{-}\left(ku_1^{+}-\frac{\partial v_1^{-}}{\partial y}\right)\right\}\cos 2z\\
&+ \frac{1}{2}\left\{\Phi_1^{-}\left(ku_1^{-}+\frac{\partial v_1^{+}}{\partial y}\right)-\Phi_1^{+}\left(ku_1^{+}-\frac{\partial v_1^{-}}{\partial y}\right)\right\}\sin 2z
\end{aligned}
$$

$$
\begin{aligned}
R_2 &= \omega_1\frac{\partial\Phi_1}{\partial z}+ku_1\frac{\partial\Phi_1}{\partial z}+v_1\frac{\partial\Phi_1}{\partial y}-\Phi_1(k\frac{\partial u_1}{\partial z}+\frac{\partial v_1}{\partial y}) =\\
&= \omega_1\left[\Phi_1^{-}\cos z-\Phi_1^{+}\sin z\right]\\
&-\frac{k}{2}\left\{u_1^{-}\Phi_1^{+}-u_1^{+}\Phi_1^{-}\right\}+\frac{k}{2}\left\{u_1^{+}\Phi_1^{-}+u_1^{-}\Phi_1^{+}\right\}\cos 2z-\frac{k}{2}\left\{u_1^{-}\Phi_1^{-}-u_1^{+}\Phi_1^{+}\right\}\sin 2z\\
&+\frac{1}{2}\left\{v_1^{+}\frac{\partial\Phi_1^{+}}{\partial y}+v_1^{-}\frac{\partial\Phi_1^{-}}{\partial y}\right\}-\frac{1}{2}\left\{v_1^{+}\frac{\partial\Phi_1^{+}}{\partial y}-v_1^{-}\frac{\partial\Phi_1^{-}}{\partial y}\right\}\cos 2z-\frac{1}{2}\left\{v_1^{+}\frac{\partial\Phi_1^{-}}{\partial y}+v_1^{-}\frac{\partial\Phi_1^{+}}{\partial y}\right\}\sin 2z\\
&-\frac{1}{2}\left\{\Phi_1^{+}\left(ku_1^{-}+\frac{\partial v_1^{+}}{\partial y}\right)-\Phi_1^{-}\left(ku_1^{+}-\frac{\partial v_1^{-}}{\partial y}\right)\right\}\\
&-\frac{1}{2}\left\{\Phi_1^{+}\left(ku_1^{-}+\frac{\partial v_1^{+}}{\partial y}\right)+\Phi_1^{-}\left(ku_1^{+}-\frac{\partial v_1^{-}}{\partial y}\right)\right\}\cos 2z\\
&-\frac{1}{2}\left\{\Phi_1^{-}\left(ku_1^{-}+\frac{\partial v_1^{+}}{\partial y}\right)-\Phi_1^{+}\left(ku_1^{+}-\frac{\partial v_1^{-}}{\partial y}\right)\right\}\sin 2z.
\end{aligned}
$$

$$
\begin{aligned}
R_3 &= \omega_1\left[\Phi_1^{-}\cos z-\Phi_1^{+}\sin z\right]\\
&- \frac{1}{2}\left\{k\left(u_1^{+}\Phi_1^{-}-u_1^{-}\Phi_1^{+}\right)+v_1^{+}\frac{\partial\Phi_1^{+}}{\partial y}+v_1^{-}\frac{\partial\Phi_1^{-}}{\partial y}\right.\\
&+ \left.\Phi_1^{+}\left(ku_1^{-}+\frac{\partial v_1^{+}}{\partial y}\right)-\Phi_1^{-}\left(ku_1^{+}-\frac{\partial v_1^{-}}{\partial y}\right)\right\}\\
&- \frac{1}{2}\left\{k\left(u_1^{+}\Phi_1^{-}+u_1^{-}\Phi_1^{+}\right)+v_1^{+}\frac{\partial\Phi_1^{+}}{\partial y}-v_1^{-}\frac{\partial\Phi_1^{-}}{\partial y}\right.\\
&+ \left.\Phi_1^{+}\left(ku_1^{-}+\frac{\partial v_1^{+}}{\partial y}\right)+\Phi_1^{-}\left(ku_1^{+}-\frac{\partial v_1^{-}}{\partial y}\right)\right\}\cos 2z\\
&- \frac{1}{2}\left\{k\left(u_1^{-}\Phi_1^{-}-u_1^{+}\Phi_1^{+}\right)+v_1^{+}\frac{\partial\Phi_1^{-}}{\partial y}+v_1^{-}\frac{\partial\Phi_1^{+}}{\partial y}\right.\\
&+ \left.\Phi_1^{-}\left(ku_1^{-}+\frac{\partial v_1^{+}}{\partial y}\right)-\Phi_1^{+}\left(ku_1^{+}-\frac{\partial v_1^{-}}{\partial y}\right)\right\}\sin 2z
\end{aligned}
$$

$$
\begin{aligned}
R_1^+(y,0) &= -\frac{1}{2}\left\{v_1^+\frac{\partial u_1^+}{\partial y} + v_1\frac{\partial u_1}{\partial y}\right\}\\
R_2^+(y,0) &= -\frac{1}{2}\left\{k\left(u_1^+v_1 - u_1v_1^+\right) + \left(v_1^+\frac{\partial v_1^+}{\partial y} + v_1\frac{\partial v_1}{\partial y}\right)\right\}\\
R_3^+(y,0) &= -\frac{1}{2}\left\{k\left(u_1^+\Phi_1 - u_1\Phi_1^+\right) + v_1^+\frac{\partial \Phi_1^+}{\partial y} + v_1\frac{\partial \Phi_1}{\partial y}\right.\\
&+ \left.\Phi_1^+\left(ku_1 + \frac{\partial v_1^+}{\partial y}\right) - \Phi_1\left(ku_1^+ - \frac{\partial v_1}{\partial y}\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
R_1^\pm(y,1) &= \pm\omega_1 u_1^\mp\\
R_2^\pm(y,1) &= \pm\omega_1 v_1^\mp\\
R_3^\pm(y,1) &= \pm\omega_1 \Phi_1^\mp
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
R_1^+(y,2) &= -\frac{1}{2}\left\{2ku_1^+u_1 + v_1^+\frac{\partial u_1^+}{\partial y} - v_1\frac{\partial u_1}{\partial y}\right\}\\
R_1(y,2) &= -\frac{1}{2}\left\{k\left[\left(u_1\right)^2 - \left(u_1^+\right)^2\right] + \left(v_1^+\frac{\partial u_1}{\partial y} + v_1\frac{\partial u_1^+}{\partial y}\right)\right\}\\
R_2^\pm(y,2) &= -\frac{1}{2}\left\{k\left(u_1^\pm v_1 \pm u_1^\mp v_1^+\right) + v_1^+\frac{\partial v_1^+}{\partial y} \mp v_1\frac{\partial v_1^+}{\partial y}\right\}\\
R_3^\pm(y,2) &= -\frac{1}{2}\left\{k\left(u_1^\pm\Phi_1 \pm u_1^\mp\Phi_1^+\right) + v_1^+\frac{\partial \Phi_1^+}{\partial y} \mp v_1\frac{\partial \Phi_1^+}{\partial y}\right.\\
&+ \left.\Phi_1^\pm\left(ku_1 + \frac{\partial v_1^+}{\partial y}\right) \pm \Phi_1^\mp\left(ku_1^+ - \frac{\partial v_1}{\partial y}\right)\right\}
\end{aligned}
$$

The right-hand-side of equation (21) is

$$
\mp\frac{\Delta_1}{\omega_0}R_2^\pm(y,1) + \frac{1}{\omega_0}y\left[\omega_0R_1^\pm(y,1) + kR_3^\pm(y,1)\right] + \frac{d}{dy}\left[kR_1^\pm(y,1) + \omega_0R_3^\pm(y,1)\right]
$$

and using (23)

$$
\omega_1\left\{-\frac{\Delta_1}{\omega_0}v_1^\mp \mp \frac{1}{\omega_0}y\left[\omega_0u_1^\mp + k\Phi_1^\mp\right] \mp \frac{d}{dy}\left[ku_1^\mp + \omega_0\Phi_1^\mp\right]\right\}
$$

$$
\begin{aligned}
\omega_0u_1^\pm + k\Phi_1^\pm &= \pm\ \omega_0\frac{A^\pm}{2\Delta_1}e^{-y^2/2}\left\{(\omega_0+k)\,H_{m+1}(y) + 2m\,(\omega_0-k)\,H_{m-1}(y)\right\}\\
&\pm\ k\frac{A^\mp}{2\Delta_1}e^{-y^2/2}\left\{(\omega_0+k)\,H_{m+1}(y) - 2m\,(\omega_0-k)\,H_{m-1}(y)\right\}\\
&= \pm\ \frac{A^\pm}{2\Delta_1}e^{-y^2/2}\left\{(\omega_0+k)^2\,H_{m+1}(y) + 2m\,(\omega_0-k)^2\,H_{m-1}(y)\right\}
\end{aligned}
$$

$$
\begin{aligned}
y\left[\omega_0 u_1^{\pm} + k\Phi_1^{\pm}\right] &= \pm \frac{A^{\mp}}{2\Delta_1} e^{-y^2/2}\left\{(\omega_0+k)^2\, y H_{m+1}(y) + 2m(\omega_0-k)^2\, y H_{m-1}(y)\right\} \\
&= \pm \frac{A^{\mp}}{2\Delta_1} e^{-y^2/2}\left\{\frac{(\omega_0+k)^2}{2} H_{m+2}(y) + (\omega_0+k)^2(m+1)H_m(y)\right. \\
&+ \left. m(\omega_0-k)^2 H_m(y) + 2m(\omega_0-k)^2(m-1)H_{m-2}(y)\right\} \\
&= \pm \frac{A^{\mp}}{2\Delta_1} e^{-y^2/2}\left\{\frac{(\omega_0+k)^2}{2} H_{m+2}(y) + \left[(\omega_0+k)^2 + 2m(k^2+\omega_0^2)\right] H_m(y)\right. \\
&+ \left. 2m(\omega_0-k)^2(m-1)H_{m-2}(y)\right\}
\end{aligned}
$$

$$
\begin{aligned}
k u_1^{\mp} + \omega_0\Phi_1^{\pm} &= \pm\, k\frac{A^{\mp}}{2\Delta_1} e^{-y^2/2}\left\{(\omega_0+k)\, H_{m+1}(y) + 2m(\omega_0-k)\, H_{m-1}(y)\right\} \\
&\pm\, \omega_0\frac{A^{\mp}}{2\Delta_1} e^{-y^2/2}\left\{(\omega_0+k)\, H_{m+1}(y) - 2m(\omega_0-k)\, H_{m-1}(y)\right\} \\
&= \pm \frac{A^{\mp}}{2\Delta_1} e^{-y^2/2}\left\{(\omega_0+k)^2\, H_{m+1}(y) - 2m(\omega_0-k)^2\, H_{m-1}(y)\right\}
\end{aligned}
$$

$$
\begin{aligned}
\mp\frac{d}{dy}\left[k u_1^{\mp} - \omega_0\Phi_1^{\mp}\right] &= \pm \frac{A^{\mp}}{2\Delta_1}\frac{d}{dy}\left\{e^{-y^2/2}\left[(\omega_0+k)^2\, H_{m+1}(y) - 2m(\omega_0-k)^2\, H_{m-1}(y)\right]\right\} \\
&= \pm \frac{A^{\mp}}{2\Delta_1} e^{-y^2/2}\left\{-y\left[(\omega_0+k)^2\, H_{m+1}(y) - 2m(\omega_0-k)^2\, H_{m-1}(y)\right]\right. \\
&+ \left. (\omega_0+k)^2\, H'_{m+1}(y) - 2m(\omega_0-k)^2\, H'_{m-1}(y)\right\} \\
&= \pm \frac{A^{\mp}}{2\Delta_1} e^{-y^2/2}\left\{-\frac{(\omega_0+k)^2}{2} H_{m+2}(y) + \left[2m(\omega_0^2+k^2) + (\omega_0+k)^2\right] H_m(y)\right. \\
&- \left. 2m(m-1)(\omega_0-k)^2\, H_{m-2}(y)\right\}
\end{aligned}
$$

Then

$$
\begin{aligned}
&-\frac{\Delta_1}{\omega_0} v_1^{\mp} \mp \frac{1}{\omega_0} y\left[\omega_0 u_1^{\pm} + k\Phi_1^{\pm}\right] \mp \frac{d}{dy}\left[k u_1^{\pm} + \omega_0\Phi_1^{\pm}\right] = \\
&-\frac{\Delta_1}{\omega_0} A^{\mp} e^{-y^2/2} H_m(y) \\
&-A^{\mp} e^{-y^2/2}\left\{\frac{(\omega_0+k)^2}{4\Delta_1\omega_0} H_{m+2}(y) + \frac{1}{2\Delta_1\omega_0}\left[(\omega_0+k)^2 + 2m(k^2+\omega_0^2)\right] H_m(y)\right. \\
&+ \left. \frac{1}{\Delta_1\omega_0} m(\omega_0-k)^2(m-1)H_{m-2}(y)\right\} \\
&-A^{\mp} e^{-y^2/2}\left\{-\frac{(\omega_0+k)^2}{4\Delta_1} H_{m+2}(y) + \frac{1}{2\Delta_1}\left[2m(\omega_0^2+k^2) + (\omega_0+k)^2\right] H_m(y)\right. \\
&- \left. \frac{1}{\Delta_1} m(m-1)(\omega_0-k)^2\, H_{m-2}(y)\right\} = \\
&-A^{\mp} e^{-y^2/2}\left\{\frac{1}{4\omega_0\Delta_1}(1-\omega_0)(\omega_0+k)^2\, H_{m+2}(y)\right. \\
&+ \frac{1}{\omega_0\Delta_1}\left\{\Delta_1^2 + \frac{1}{2}(1+\omega_0)(\omega_0+k)^2 + (1+\omega_0)\, m(k^2+\omega_0^2)\right\} H_m(y)
\end{aligned}
$$

$$+\frac{1}{\omega_0\Delta_1}m(m-1)(1-\omega_0)(\omega_0-k)^2 H_{m-2}(y)\Big\}$$

$$R_1^+(y,0) = -\frac{1}{2}\left\{v_1^+\frac{\partial u_1^+}{\partial y} + v_1^-\frac{\partial u_1^-}{\partial y}\right\} =$$

$$-\frac{1}{2}\left[A^+e^{-y^2/2}H_m(y)\right]\frac{\partial}{\partial y}\left\{\frac{A^-}{2\Delta_1}e^{-y^2/2}\left\{(\omega_0+k)H_{m+1}(y)+2m(\omega_0-k)H_{m-1}(y)\right\}\right\}$$

$$-\frac{1}{2}A^-e^{-y^2/2}H_m(y)\frac{\partial}{\partial y}\left\{-\frac{A^+}{2\Delta_1}e^{-y^2/2}\left\{(\omega_0+k)H_{m+1}(y)+2m(\omega_0-k)H_{m-1}(y)\right\}\right\} =$$

$$-\frac{A^+A^-}{4\Delta_1}e^{-y^2/2}H_m(y)\frac{\partial}{\partial y}\left\{e^{-y^2/2}\left\{(\omega_0+k)H_{m+1}(y)+2m(\omega_0-k)H_{m-1}(y)\right\}\right\}$$

$$+\frac{A^-A^+}{4\Delta_1}e^{-y^2/2}H_m(y)\frac{\partial}{\partial y}\left\{e^{-y^2/2}\left\{(\omega_0+k)H_{m+1}(y)+2m(\omega_0-k)H_{m-1}(y)\right\}\right\} = 0$$

$$R_2^+(y,0) = -\frac{1}{2}\left\{k\left(u_1^+v_1^- - u_1^-v_1^+\right) + v_1^+\frac{\partial v_1^+}{\partial y} + v_1^-\frac{\partial v_1^-}{\partial y}\right\}$$

$$u_1^+v_1^- = \left[\frac{A^-}{2\Delta_1}e^{-y^2/2}\left\{(\omega_0+k)H_{m+1}(y)+2m(\omega_0-k)H_{m-1}(y)\right\}\right]A^-e^{-y^2/2}H_m(y) =$$

$$\frac{(A^-)^2}{2\Delta_1}e^{-y^2}H_m(y)\left[(\omega_0+k)H_{m+1}(y)+2m(\omega_0-k)H_{m-1}(y)\right]$$

$$u_1^-v_1^+ = -\frac{A^+}{2\Delta_1}e^{-y^2/2}\left\{(\omega_0+k)H_{m+1}(y)+2m(\omega_0-k)H_{m-1}(y)\right\}A^+e^{-y^2/2}H_m(y) =$$

$$-\frac{(A^+)^2}{2\Delta_1}e^{-y^2}\left[(\omega_0+k)H_{m+1}(y)+2m(\omega_0-k)H_{m-1}(y)\right]$$

$$v_1^+\frac{\partial v_1^+}{\partial y} = (A^+)^2e^{-y^2/2}H_m(y)\frac{\partial}{\partial y}\left[e^{-y^2/2}H_m(y)\right]$$

$$v_1^-\frac{\partial v_1^-}{\partial y} = (A^-)^2e^{-y^2/2}H_m(y)\frac{\partial}{\partial y}\left[e^{-y^2/2}H_m(y)\right]$$

$$k\left(u_1^+v_1^- - u_1^-v_1^+\right) + v_1^+\frac{\partial v_1^+}{\partial y} + v_1^-\frac{\partial v_1^-}{\partial y}$$

$$R_2^+(y,0) = -\frac{k}{2}\left[\frac{(A^+)^2+(A^-)^2}{2\Delta_1}\right]e^{-y^2}H_m(y)\left[(\omega_0+k)H_{m+1}(y)+2m(\omega_0-k)H_{m-1}(y)\right]+$$

$$\frac{1}{2}\left[(A^+)^2+(A^-)^2\right]e^{-y^2/2}H_m(y)\frac{\partial}{\partial y}\left[e^{-y^2/2}H_m(y)\right] =$$

$$-\frac{k}{2}\left[\frac{(A^+)^2+(A^-)^2}{2\Delta_1}\right]e^{-y^2}H_m(y)\left[(\omega_0+k)H_{m+1}(y)+2m(\omega_0-k)H_{m-1}(y)\right]+$$

$$\frac{1}{2}\left[(A^+)^2+(A^-)^2\right]e^{-y^2}H_m(y)\left[2mH_{m-1}(y)-yH_m(y)\right] -$$

$$-\frac{k}{2}\left[\frac{(A^+)^2+(A^-)^2}{2\Delta_1}\right]e^{-y^2}H_m(y)\left[(\omega_0+k)H_{m+1}(y)+2m(\omega_0-k)H_{m-1}(y)\right]+$$

$$\frac{1}{2}\left[(A^+)^2+(A^-)^2\right]e^{-y^2}H_m(y)\left\{2mH_{m-1}(y)-\frac{1}{2}H_{m+1}(y)-mH_{m-1}(y)\right\} =$$

$$-\frac{k}{2}\left[\frac{(A^+)^2+(A^-)^2}{2\Delta_1}\right]e^{-y^2}H_m(y)\left[(\omega_0+k)H_{m+1}(y)+2m(\omega_0-k)H_{m-1}(y)\right]+$$

$$\frac{1}{2}\left[(A^+)^2+(A^-)^2\right]e^{-y^2}H_m(y)\left\{mH_{m-1}(y)-\frac{1}{2}H_{m+1}(y)\right\}$$

$$\frac{\omega_0\left[(A^+)^2+(A^-)^2\right]}{2(\omega_0-k)}e^{-y^2}H_m(y)\left\{mH_{m-1}(y)-\frac{1}{2}H_{m+1}(y)\right\}$$

$$R_2^+(y,0)=\frac{\omega_0\left[(A^+)^2+(A^-)^2\right]}{2(\omega_0-k)}e^{-y^2}H_m(y)\left\{mH_{m-1}(y)-\frac{1}{2}H_{m+1}(y)\right\}$$

$$R_3^+(y,0)=-\frac{1}{2}\left\{k\left(u_1^+\Phi_1-u_1\Phi_1^+\right)+v_1^+\frac{\partial\Phi_1^+}{\partial y}+v_1\frac{\partial\Phi_1}{\partial y}\right.$$
$$\left.+\Phi_1^+\left(ku_1+\frac{\partial v_1^+}{\partial y}\right)-\Phi_1\left(ku_1^+-\frac{\partial v_1}{\partial y}\right)\right\}$$

$$v_1^+\Phi_1=-\frac{A^+A}{4\Delta_1^2}e^{-y^2}\left\{(\omega_0+k)^2H_{m+1}^2(y)-4m^2(\omega_0-k)^2H_{m-1}^2(y)\right\}$$
$$\Phi_1^+=-\frac{A^+A}{4\Delta_1^2}e^{-y^2}\left\{(\omega_0+k)^2H_{m+1}^2(y)-4m^2(\omega_0-k)^2H_{m-1}^2(y)\right\}$$
$$\frac{\partial\Phi_1^+}{\partial y}=\frac{A^+A}{2\Delta_1}e^{-y^2/2}H_m(y)\frac{\partial}{\partial y}e^{-y^2/2}\left\{(\omega_0+k)H_{m+1}(y)-2m(\omega_0-k)H_{m-1}(y)\right\}$$
$$\frac{\partial\Phi_1}{\partial y}=\frac{A^+A}{2\Delta_1}e^{-y^2/2}H_m(y)\frac{\partial}{\partial y}e^{-y^2/2}\left\{(\omega_0+k)H_{m+1}(y)-2m(\omega_0-k)H_{m-1}(y)\right\}$$
$$-\frac{\partial v_1^+}{\partial y}=A^+\left\{\frac{\partial}{\partial y}e^{-y^2/2}H_m(y)-\frac{k}{2\Delta_1}e^{-y^2/2}\left[(\omega_0+k)H_{m+1}(y)+2m(\omega_0-k)H_{m-1}(y)\right]\right\}$$
$$-\frac{\partial v_1}{\partial y}=-A\left\{\frac{\partial}{\partial y}A\,e^{-y^2/2}H_m(y)-\frac{k}{2\Delta_1}e^{-y^2/2}\left[(\omega_0+k)H_{m+1}(y)+2m(\omega_0-k)H_{m-1}(y)\right.\right.$$
$$-\frac{1}{2}\Phi_1^+\left(ku_1-\frac{\partial v_1^+}{\partial y}\right)+\frac{1}{2}\Phi_1^-\left(ku_1^+-\frac{\partial v_1}{\partial y}\right)$$
$$\frac{1}{2}A\;A^+-A^+(-A)=0$$

3.6.1. Summary.

$$R_1^-(y,0)=0$$
$$R_2^-(y,0)=\frac{\omega_0\left[(A^+)^2+(A^-)^2\right]}{2(\omega_0-k)}e^{-y^2}H_m(y)\left\{mH_{m-1}(y)-\frac{1}{2}H_{m+1}(y)\right\}$$
$$R_3^+(y,0)=0$$

Then from the equations in page 5

$$v_2^+(y,0)=0$$

In order study the problem in general, one must observe the following facts:

1. We have already determined that when $n=1$, (21) is an homogeneous equations with solution given by $p=1$ and

$$v_1^+(y,1)=e^{-\frac{y^2}{2}}H_m(y)$$

where $H_m(y)$ is the Hermite polynomial of degree m, provided $\omega_0^2 - \frac{k}{\omega_0} - k^2 = 2m+1$.

1. The right-hand-sides in (14) - and therefore the right hand-side of equation (21) - have two types of terms: linear in the unknown functions (multiplied by ω_j) and quadratic. These are, for example, of the forms

$$\sum_{j=1}^{n-1} u_j \frac{\partial u_{n-j}}{\partial x}.$$

   Thus, when $n=2$ the product yields a factor $\exp(-2(y^2/2))$ and the solution must contain terms proportional to it. For $n=3$ this right-hand-side has the products $u_1 \frac{\partial u_2}{\partial x}$ and $u_2 \frac{\partial u_1}{\partial x}$ and therefore the factor $\exp(-3(y^2/2))$. In general, if u_j contains terms proportional to $\exp(-j(y^2/2))$, the product will have a factor $\exp(-n(y^2/2))$.
2. By the same token, these products yield harmonics of order at most of the same order of the approximation.

Let us denote with $R^\pm(y,p)$ the right-hand-side of the (21) and with $R^\pm_q(y,p)$ the coefficient of $e^{-q\frac{y^2}{2}}$ in it.

Using the linearity of equation (21) and the preceeding conclusions, we can write

$$v_n^\pm(y,p) = \sum_{q=1}^{n} e^{-q\frac{y^2}{2}} v^\pm_{n,q}(y,p).$$

Then:

$$\frac{dv_n^\pm(y,p)}{dy} = \sum_{q=1}^{n} \left\{ -qyv^\pm_{n,q}(y,p) + \frac{dv^\pm_{n,q}(y,p)}{dy} \right\} e^{-q\frac{y^2}{2}}$$

$$\sum_{j=1}^{n} \frac{d^2 v^\mp_{n,q}(y,p)}{dy^2} = \sum_{q=1}^{n} \left\{ (q^2y^2 - q)\, v^\pm_{n,q}(y,p) - 2qy\frac{dv^\pm_{n,q}(y,p)}{dy} + \frac{d^2 v^\pm_{n,q}(y,p)}{dy^2} \right\} e^{-q\frac{y^2}{2}}$$

$$\frac{d^2 v_n^\mp(y,p)}{dy^2} + \left\{ p^2\Delta_1 - \frac{k}{\omega_0} - y^2 \right\} v_n^\mp(y,p) =$$

$$\sum_{q=1}^{n} \left\{ \frac{d^2 v^\mp_{n,q}(y,p)}{dy^2} - 2qy\frac{dv^\pm_{n,q}(y,p)}{dy} + \left[p^2\Delta_1 - \frac{k}{\omega_0} - q - (1-q^2)\, y^2 \right] v^\pm_{n,q}(y,p) \right\} e^{-q\frac{y^2}{2}}.$$

Therefore:

$$\frac{d^2 v^\mp_{n,q}(y,p)}{dy^2} - 2qy\frac{dv^\pm_{n,q}(y,p)}{dy} + \left[p^2\Delta_1 - \frac{k}{\omega_0} - q - \left(1-q^2\right) y^2 \right] v^\pm_{n,q}(y,p) = R^\pm_q(y,p)$$

Let us expand $R^\pm_q(y,p)$ in Hermite polynomials as

$$R^\pm_q(y,p) = \sum_{r=0}^{\infty} R^\pm_q(r,p) H_r(y)$$

and then we can assume

$$v^{+}_{n,q}(y,p) = \sum_{r=0}^{\infty} \hat{v}^{+}_{n,q}(r,p) H_r(y).$$

Observing that

$$\frac{d^2 H_r(y)}{dy^2} - 2qy\frac{dH_r(y)}{dy} + \left[p^2\Delta_1 - \frac{k}{\omega_0} - q - (1-q^2)\,y^2\right] H_r(y) =$$

$$2(1-q)yH_r'(y) + \left[p^2\Delta_1 - \frac{k}{\omega_0} - q - (1-q^2)\,y^2 - 2r\right] H_r(y) =$$

$$4r(1-q)yH_{r-1}(y) + \left[p^2\Delta_1 - \frac{k}{\omega_0} - q - 2r\right] H_r(y) - (1-q^2)\,y^2 H_r(y) =$$

$$4r(1-q)\left[\frac{1}{2}H_r(y) + (r-1)\,H_{r-2}(y)\right] + \left[p^2\Delta_1 - \frac{k}{\omega_0} - q - 2r\right] H_r(y)$$

$$-(1-q^2)\,y\left[\frac{1}{2}H_{r+1}(y) + rH_{r-1}(y)\right]$$

$$4r(r-1)(1-q)H_{r-2}(y) + \left[p^2\Delta_1 - \frac{k}{\omega_0} - q - 2r + 2r(1-q)\right] H_r(y)$$

$$-\frac{1}{2}(1-q^2)\left[\frac{1}{2}H_{r+2}(y) + (r+1)\,H_r(y)\right] - (1-q^2)\,r\left[\frac{1}{2}H_r(y) + (r-1)\,H_{r-2}(y)\right] =$$

$$\{4r(r-1)(1-q) - (1-q^2)\,r\,(r-1)\}\,H_{r-2}(y) +$$

$$\left\{p^2\Delta_1 - \frac{k}{\omega_0} - q - 2r + 2r(1-q) - \frac{1}{2}(1-q^2)(r+1) - \frac{1}{2}(1-q^2)\,r\right\} H_r(y) +$$

$$\left.-\frac{1}{4}(1-q^2)\right\} H_{r+2}(y) =$$

$$r(r-1)(1-q)(3-q)\,H_{r-2}(y) +$$

$$\left\{p^2\Delta_1 - \frac{k}{\omega_0} - q - 2rq - \frac{1}{2}(1-q^2)(2r+1)\right\} H_r(y)$$

$$-\frac{1}{4}(1-q^2)\,H_{r+2}(y) =$$

$$r(r-1)(1-q)(3-q)\,H_{r-2}(y) +$$

$$\left\{(p^2-1)\,\Delta_1 + 2\,(m - rq) + (1-q)\left[\frac{1}{2}(1-q) - r(1+q)\right]\right\} H_r(y)$$

$$-\frac{1}{4}(1-q^2)\,H_{r+2}(y).$$

then:

$$\sum_{r=2}^{\infty} r\,(r-1)\,(1-q)\,(3-q)\,\hat{v}^{\pm}_{n,q}(r,p) H_{r-2}(y) +$$

$$\sum_{r=0}^{\infty} \left\{(p^2-1)\,\Delta_1 + 2\,(m - rq) + (1-q)\left[\frac{1}{2}(1-q) - r(1+q)\right]\right\} \hat{v}^{+}_{n,q}(r,p)H_r(y)$$

$$-\frac{1}{4}(1-q^2) \sum_{r=0}^{\infty} \hat{v}^{+}_{n,q}(r,p)H_{r+2}(y) =$$

$$(1-q)\,(3-q) \sum_{r=0}^{\infty} (r+2)\,(r+1)\,\hat{v}^{+}_{n,q}(r+2,p)H_r(y) +$$

$$\sum_{r=0}^{\infty} \left\{(p^2-1)\,\Delta_1 + 2\,(m - rq) + (1-q)\left[\frac{1}{2}(1-q) - r(1+q)\right]\right\} \hat{v}^{+}_{n,q}(r,p)H_r(y)$$

$$-\frac{1}{4}(1-q^2) \sum_{r=2}^{\infty} \hat{v}^{+}_{n,q}(r-2,p)H_r(y) =$$

$$\left\{\left[(p^2-1)\,\Delta_1+2m+\frac{1}{2}\,(1-q)^2\right]v^{\pm}_{n,q}(0,p)+2(1-q)\,(3-q)\,v^{\pm}_{n,q}(2,p)\right\}H_0(y)+$$
$$\left\{6(1-q)\,(3-q)\,v^{\pm}_{n,q}(3,p)+\left[(p^2-1)\,\Delta_1+2m+2-\frac{5q}{2}-\frac{3}{2}q^2\right]v^{\pm}_{n,q}(1,p)\right\}H_1(y)+$$
$$\sum_{r\geq 2}\left\{(1-q)\,(3-q)\sum_{r=0}(r+2)\,(r+1)\,v^{\pm}_{n,q}(r+2,p)+\right.$$
$$\left[(p^2-1)\,\Delta_1+2\,(m-rq)+(1-q)\left[\frac{1}{2}\,(1-q)-r(1+q)\right]\right]v_{n,q}(r,p)$$
$$\left.-\frac{1}{4}\,(1-q^2)\,v^{\pm}_{n,q}(r-2,p)\right\}H_r(y).$$

Hence:

$$\left[(p^2-1)\,\Delta_1+2m+\frac{1}{2}\,(1-q)^2\right]\hat{v}^{\pm}_{n,q}(0,p)+2(q-1)\,(q-3)\,v^{\pm}_{n,q}(2,p)=\hat{R}^{\pm}_{,q}(0,p),$$

$$\left[(p^2-1)\,\Delta_1+2\,(m-1)+\frac{3}{2}\,(q-1)^2\right]v^{\pm}_{n,q}(1,p)+6(q-1)\,(q-3)\,\hat{v}^{\pm}_{n,q}(3,p)=\hat{R}^{\pm}_{,q}(1,p),$$

$$\begin{aligned}
&-\frac{1}{4}\,(1-q^2)\,v^{\pm}_{n,q}(r-2,p) &+\\
\left\{(p^2-1)\,\Delta_1+2\,(m-r)+\frac{1}{2}\,(1-q)\,(1+2r)\right\}&v^{\pm}_{n,q}(r,p) &+\\
(q-1)\,(q-3)\,(r+2)\,(r+1)\,&\hat{v}^{\pm}_{n,q}(r+2,p) &= \quad \hat{R}^{\pm}_{,q}(r,p).
\end{aligned}$$

Observe that at any given order of approximation n we have por each p between 0 [illegible] and any q from 1 to n the system of equations

$$M\hat{v}^{\pm}_{n,q}(\cdot,p)=\hat{R}^{\pm}_{,q}(\cdot,p)$$

[illegible] M is the three-diagonal matrix defined by the coefficients

$$\begin{aligned}
M_{r,r-2} &= -\frac{1}{4}\,(1-q^2)\,,\\
M_{r,r} &= (p^2-1)\,\Delta_1+2\,(m-r)+\frac{1}{2}\,(1-q)\,(1+2r)\,,\\
M_{r,r+2} &= (q-1)\,(q-3)\,(r+2)\,(r+1)\,.
\end{aligned}$$

In order to solve the differential equation, we start each step choosing $p=1,q=1$,then [illegible] is reduced to

$$\begin{pmatrix}
2m & 0 & 0 & 0 & \cdots\\
0 & 2\,(m-1) & 0 & 0 & \cdots\\
\vdots & \cdots & \cdots & & \vdots\\
0 & 0 & 2\,(m-r) & 0 & \cdots\\
 & & \vdots & &
\end{pmatrix}
\begin{pmatrix}
\hat{v}^{\pm}_{n,q}(0,1)\\
v^{\pm}_{n,q}(1,1)\\
\vdots\\
v^{\pm}_{n,q}(r,1)\\
\vdots
\end{pmatrix}
=
\begin{pmatrix}
\hat{R}^{\pm}_{,q}(0,1)\\
\hat{R}^{\pm}_{,q}(1,1)\\
\vdots\\
\hat{R}^{\pm}_{,q}(r,1)\\
\vdots
\end{pmatrix}.$$

The matrix is diagonal with only one line zero, namely the $m-th$ one. The system will have a solution if and only if

$$\hat{R}^{\pm}_{,q}(r,1)=0, \tag{24}$$

which is a linear equation in ω_{n-1}.We choose it in such a way as to satisfy (24) and use this result in the other equations. For all other values of p, the corresponding system can simply be solved.

4. Discussion of Results. We have determined a constant solution as well as an exponentially decaying one, which is independent of x and t. We have also found travelling-wave soltuions of the variational equations around the trivial solution. However, several problems still remain to be studied. The existence of solutions of the variational equations does not imply by itself that the non-linear problem has a solution of this type. In fact, it would be necessary to prove that the solvability condition, that is, the coefficient of ω_1 in the second order approximation, is different from zero, a thing which is not obvious at all. It would also be important to analyze what happens in the case of coalescence, i.e., the discrete set of values of ω and k for which more than one Hermite polynomial satisfy the variational equations. Finally, it would be necessary to analytically expand the right-hand-sides of the equations in order to be able to establish an infinite system of equations which determins the Fourier coefficients and numerically solve a truncated system.

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6. Appendix A.

6.1. RHS.

FIRST EQUATION

$$-\omega_j \frac{\partial u_{n-j}}{\partial z} + k u_j \frac{\partial u_{n-j}}{\partial z} + v_j \frac{\partial u_{n-j}}{\partial y}$$

First Term

$\omega_j \sum_{p=1}^{\infty} p \left[-u_{n-j}^{-}(y,p) \cos pz + u_{n-j}^{+}(y,p) \sin pz \right]$

Second Term

$\sum_{p=0}^{\infty} \left[u_j^{+}(y,p) \cos pz + u_j^{-}(y,p) \sin pz \right] \sum_{p=1}^{\infty} p \left[u_{n-j}^{-}(y,p) \cos pz - u_{n-j}^{+}(y,p) \sin pz \right] =$

$\frac{1}{2} \sum_{q=1}^{\infty} q \left[u_j^{+}(y,q) u_{n-j}^{-}(y,q) - u_j^{-}(y,q) u_{n-j}^{+}(y,q) \right] +$

$\sum_{p=1}^{\infty} \Big\{ p u_j^{+}(y,0) u_{n-j}^{-}(y,p) +$

$\frac{1}{2} \sum_{q=1}^{\infty} \Big[q \left(u_j^{+}(y,p+q) u_{n-j}^{-}(y,q) - u_j^{-}(y,p+q) u_{n-j}^{+}(y,q) \right) +$

$(p-q) \left(u_j^{+}(y,q) u_{n-j}^{-}(y,p+q) - u_j^{-}(y,q) u_{n-j}^{+}(y,p+q) \right) \Big] +$

$\frac{1}{2} \sum_{q=1}^{p-1} (p-q) \left[u_j^{+}(y,q) u_{n-j}^{-}(y,p-q) + u_j^{-}(y,q) u_{n-j}^{+}(y,p-q) \right] \Big\} \cos pz +$

$\sum_{p=1}^{\infty} \Big\{ -u_j^{+}(y,0) u_{n-j}^{+}(y,p) +$

$\frac{1}{2} \sum_{q=1}^{\infty} \Big[q \left(u_j^{-}(y,p+q) u_{n-j}^{-}(y,q) + u_j^{+}(y,p+q) u_{n-j}^{+}(y,q) \right) -$

$(p-q) \left(u_j^{+}(y,q) u_{n-j}^{+}(y,p+q) + u_j^{-}(y,q) u_{n-j}^{-}(y,p+q) \right) \Big] +$

$\frac{1}{2} \sum_{q=1}^{p-1} (p-q) \left[u_j^{-}(y,q) u_{n-j}^{-}(y,p-q) - u_j^{+}(y,q) u_{n-j}^{+}(y,p-q) \right] \Big\} \sin pz =$

$\sum_{p=1}^{\infty} \Big\{ p u_j^{-}(y,0) u_{n-j}^{-}(y,p) +$

$\frac{1}{2} \sum_{q=1}^{\infty} p \left[u_j^{+}(y,q) u_{n-j}^{-}(y,p+q) - u_j^{-}(y,q) u_{n-j}^{+}(y,p+q) \right] +$

$\frac{1}{2} \sum_{q=1}^{p-1} (p-q) \left[u_j^{+}(y,q) u_{n-j}^{-}(y,p-q) + u_j^{-}(y,q) u_{n-j}^{+}(y,p-q) \right] \Big\} \cos pz +$

$\sum_{p=1}^{\infty} \Big\{ -u_j^{+}(y,0) u_{n-j}^{+}(y,p) - \frac{1}{2} \sum_{q=1}^{\infty} p \left[u_j^{+}(y,q) u_{n-j}^{+}(y,p+q) + u_j^{-}(y,q) u_{n-j}^{-}(y,p+q) \right] +$

$\frac{1}{2} \sum_{q=1}^{p-1} (p-q) \left[u_j^{-}(y,q) u_{n-j}^{-}(y,p-q) - u_j^{+}(y,q) u_{n-j}^{+}(y,p-q) \right] \Big\} \sin pz$

Third Term

$\sum_{p=0}^{\infty} \left[v_j^{+}(y,p) \cos pz + v_j^{-}(y,p) \sin pz \right] \sum_{p=0}^{\infty} \left[\frac{du_{n-j}^{+}(y,p)}{dy} \cos pz + \frac{du_{n-j}^{-}(y,p)}{dy} \sin pz \right] =$

$$v_j^+(y,0)\frac{du^+_{n-j}(y,0)}{dy} + \frac{1}{2}\sum_{q=1}^{\infty}\left[v_j^+(y,q)\frac{du^+_{n-j}(y,q)}{dy} + v_j^-(y,q)\frac{du^-_{n-j}(y,q)}{dy}\right] +$$

$$\sum_{p=1}^{\infty}\left\{v_j^+(y,0)\frac{du^+_{n-j}(y,p)}{dy} + v_j^+(y,p)\frac{du^+_{n-j}(y,0)}{dy} + \right.$$

$$\frac{1}{2}\sum_{q=1}^{\infty}\left[v_j^+(y,p+q)\frac{du^+_{n-j}(y,q)}{dy} + v_j^-(y,p+q)\frac{du^-_{n-j}(y,q)}{dy} + \right.$$

$$\left. v_j^+(y,q)\frac{du^+_{n-j}(y,p+q)}{dy} + v_j^-(y,q)\frac{du^-_{n-j}(y,p+q)}{dy}\right] +$$

$$\left.\frac{1}{2}\sum_{q=1}^{p-1}\left[v_j^+(y,q)\frac{du^+_{n-j}(y,p-q)}{dy} - v_j^-(y,q)\frac{du^-_{n-j}(y,p-q)}{dy}\right]\right\}\cos pz +$$

$$\sum_{p=1}^{\infty}\left\{v_j^+(y,0)\frac{du^-_{n-j}(y,p)}{dy} + v_j^-(y,p)\frac{du^+_{n-j}(y,0)}{dy} + \right.$$

$$\frac{1}{2}\sum_{q=1}^{\infty}\left[v_j^+(y,q)\frac{du^-_{n-j}(y,p+q)}{dy} + v_j^-(y,p+q)\frac{du^+_{n-j}(y,q)}{dy}\right.$$

$$\left. - v_j^+(y,p+q)\frac{du^-_{n-j}(y,q)}{dy} - v_j^-(y,q)\frac{du^+_{n-j}(y,p+q)}{dy}\right] +$$

$$\left.\frac{1}{2}\sum_{q=1}^{p-1}\left[v_j^+(y,q)\frac{du^-_{n-j}(y,p-q)}{dy} + v_j^-(y,q)\frac{du^+_{n-j}(y,p-q)}{dy}\right]\right\}\sin pz$$

zero terms

$$v_j^-(y,0)\frac{du^-_{n-j}(y,0)}{dy} +$$

$$\frac{1}{2}\sum_{q=1}^{\infty}\left\{qk\left[u_j^+(y,q)u^-_{n-j}(y,q) - u_j^-(y,q)u^+_{n-j}(y,q)\right] + \left[v_j^+(y,q)\frac{du^+_{n-j}(y,q)}{dy} + v_j^-(y,q)\frac{du^-_{n-j}(y,q)}{dy}\right.\right.$$

cosine equation

$$-p\omega u^-_{n-j}(y,p) + pku_j^+(y,0)u^-_{n-j}(y,p) + v_j^+(y,0)\frac{du^+_{n-j}(y,p)}{dy} + v_j^+(y,p)\frac{du^+_{n-j}(y,0)}{dy} +$$

$$\frac{1}{2}\sum_{q=1}^{\infty}\left\{kp\left[u_j^+(y,q)u^-_{n-j}(y,p+q) - u_j^-(y,q)u^+_{n-j}(y,p+q)\right] + \right.$$

$$v_j^+(y,p+q)\frac{du^+_{n-j}(y,q)}{dy} + v_j^-(y,p+q)\frac{du^-_{n-j}(y,q)}{dy} +$$

$$\left.\left. v_j^+(y,q)\frac{du^+_{n-j}(y,p+q)}{dy} + v_j^-(y,q)\frac{du^-_{n-j}(y,p+q)}{dy}\right]\right\} +$$

$$\frac{1}{2}\sum_{q=1}^{p-1}\left\{k(p-q)\left[u_j^+(y,q)u^-_{n-j}(y,p-q) + u_j^-(y,q)u^+_{n-j}(y,p-q)\right] + \right.$$

$$\left.\left. v_j^+(y,q)\frac{du^+_{n-j}(y,p-q)}{dy} - v_j^-(y,q)\frac{du^-_{n-j}(y,p-q)}{dy}\right]\right\}$$

sine equation

$$-pu^+_{n-j}(y,p) - ku_j^+(y,0)u^+_{n-j}(y,p) + v_j^+(y,0)\frac{du^-_{n-j}(y,p)}{dy} + v_j^-(y,p)\frac{du^+_{n-j}(y,0)}{dy}$$

$$-\frac{1}{2}\sum_{q=1}^{\infty}\left\{pk\left[u_j^+(y,q)u^+_{n-j}(y,p+q) + u_j^-(y,q)u^-_{n-j}(y,p+q)\right] - \right.$$

$$v_j^+(y,q)\frac{du^-_{n-j}(y,p+q)}{dy} + v_j^-(y,p+q)\frac{du^+_{n-j}(y,q)}{dy}$$

$$-v_j^+(y,p+q)\frac{du_{n-j}(y,q)}{dy} - v_j^-(y,q)\frac{du_{n-j}^+(y,p+q)}{dy}\Bigg]\Bigg\}+$$

$$\frac{1}{2}\sum_{q=1}^{p-1}\Big\{k(p-q)\Big[u_j^-(y,q)u_{n-j}(y,p-q) - u_j^+(y,q)u_{n-j}^+(y,p-q)\Big]+$$

$$\Bigg[v_j^+(y,q)\frac{du_{n-j}(y,p-q)}{dy} + v_j^-(y,q)\frac{du_{n-j}^+(y,p-q)}{dy}\Bigg]\Bigg\}$$

joint equations

$$\mp p\omega_j u_{n-j}^{\pm}(y,p) \pm pku_j^{\pm}(y,0)u_{n-j}^{\pm}(y,p) + v_j^{\pm}(y,0)\frac{du_{n-j}^{\pm}(y,p)}{dy} + v_j^{\pm}(y,p)\frac{du_{n-j}^{\pm}(y,0)}{dy}$$

$$\pm\frac{1}{2}\sum_{q=1}^{\infty}\Big\{kp\Big[u_j^{\mp}(y,q)u_{n-j}^{\pm}(y,p+q) \mp u_j(y,q)u_{n-j}^{\pm}(y,p+q)\Big]+$$

$$\Bigg[v_j^+(y,p+q)\frac{du_{n-j}^{\pm}(y,q)}{dy} \pm v_j(y,p+q)\frac{du_{n-j}^{\mp}(y,q)}{dy} \pm$$

$$v_j^{\mp}(y,q)\frac{du_{n-j}^{\pm}(y,p+q)}{dy} + v_j^-(y,q)\frac{du_{n-j}^{\mp}(y,p+q)}{dy}\Bigg]\Bigg\}+$$

$$\frac{1}{2}\sum_{q=1}^{p-1}\Big\{k(p-q)\Big[u_j^{\pm}(y,q)u_{n-j}(y,p-q) \pm u_j^{\mp}(y,q)u_{n-j}^{\pm}(y,p-q)\Big]+$$

$$\Bigg[v_j(y,q)\frac{du_{n-j}^{\pm}(y,p-q)}{dy} \mp v_j(y,q)\frac{du_{n-j}^{\pm}(y,p-q)}{dy}\Bigg]\Bigg\}$$

SECOND EQUATION

$$-\sum_{j=1}^{n-1}\left\{-\omega_j\frac{\partial v_{n-j}}{\partial z} + ku_j\frac{\partial v_{n-j}}{\partial z} + v_j\frac{\partial v_{n-j}}{\partial y}\right\}$$

First term

$$-\sum_{p=1}^{\infty} p\left[-v_n^-(y,p)\cos pz + v_n^+(y,p)\sin pz\right]$$

Second term

$$\sum_{p=1}^{\infty}\left[u_j^+(y,p)\cos pz + u_j^-(y,p)\sin pz\right]\sum_{p=1}^{\infty}p\left[v_{n-j}^-(y,p)\cos pz - v_{n-j}^+(y,p)\sin pz\right] =$$

$$\frac{1}{2}\sum_{j=1}^{\infty}q\left[u_j^+(y,q)v_{n-j}^-(y,q) - u_j^-(y,q)v_{n-j}^+(y,q)\right]+$$

$$\sum_{p=1}^{\infty}\Big\{u_j^+(y,0)v_{n-j}^-(y,p)+$$

$$\frac{1}{2}\sum_{q=1}^{\infty}\Big[q\left(u_j^+(y,p+q)v_{n-j}^-(y,q) - u_j^-(y,p+q)v_{n-j}^+(y,q)\right)+$$

$$(p+q)\left(u_j^+(y,q)v_{n-j}^-(y,p+q) - u_j^-(y,q)v_{n-j}^+(y,p+q)\right)\Big]+$$

$$\frac{1}{2}\sum_{q=1}^{p-1}(p-q)\Big[u_j^+(y,q)v_{n-j}^-(y,p-q) + u_j^-(y,q)v_{n-j}^+(y,p-q)\Big]\Bigg\}\cos pz+$$

$$\sum_{p=1}^{\infty}\Big\{-u_j^+(y,0)v_{n-j}^+(y,p)+$$

$$\frac{1}{2}\sum_{q=1}^{\infty}\Big[q\left(u_j^-(y,p+q)v_{n-j}^-(y,q) + u_j^+(y,p+q)v_{n-j}^+(y,q)\right)-$$

$$(p+q)\left(u_j^+(y,q)v_{n-j}^+(y,p+q)+u_j^-(y,q)v_{n-j}^-(y,p+q)\right)\Big| +$$

$$\frac{1}{2}\sum_{q=1}^{p-1}(p-q)\left(u_j^-(y,q)v_{n-j}^-(y,p-q)-u_j^+(y,q)v_{n-j}^+(y,p-q)\right)\Big\}\sin pz$$

Third term

$$\sum_{p=0}^{\infty}\left[v_j^+(y,p)\cos pz+v_j^-(y,p)\sin pz\right]\sum_{p=0}^{\infty}\left[\frac{dv_{n-j}^+(y,p)}{dy}\cos pz+\frac{dv_{n-j}^-(y,p)}{dy}\sin pz\right]=$$

$$v_j^+(y,0)\frac{dv_{n-j}^+(y,0)}{dy}+\frac{1}{2}\sum_{q=1}^{\infty}\left[v_j^+(y,q)\frac{dv_{n-j}^+(y,q)}{dy}+v_j^-(y,q)\frac{dv_{n-j}^-(y,q)}{dy}\right]+$$

$$\sum_{p=1}^{\infty}\Big\{v_j^+(y,0)\frac{dv_{n-j}^+(y,p)}{dy}+v_j^+(y,p)\frac{dv_{n-j}^+(y,0)}{dy}+$$

$$\frac{1}{2}\sum_{q=1}^{\infty}\left[v_j^+(y,p+q)\frac{dv_{n-j}^+(y,q)}{dy}+v_j^-(y,p+q)\frac{dv_{n-j}^-(y,q)}{dy}+\right.$$

$$\left.v_j^+(y,q)\frac{dv_{n-j}^+(y,p+q)}{dy}+v_j^-(y,q)\frac{dv_{n-j}^-(y,p+q)}{dy}\right]+$$

$$\frac{1}{2}\sum_{q=1}^{p-1}\left[v_j^+(y,q)\frac{dv_{n-j}^+(y,p-q)}{dy}-v_j^-(y,q)\frac{dv_{n-j}^-(y,p-q)}{dy}\right]\Big\}\cos pz+$$

$$\sum_{p=1}^{\infty}\Big\{v_j^-(y,0)\frac{dv_{n-j}^-(y,p)}{dy}+v_j^-(y,p)\frac{dv_{n-j}^+(y,0)}{dy}+$$

$$\frac{1}{2}\sum_{q=1}^{\infty}\left[v_j^+(y,q)\frac{dv_{n-j}^-(y,p+q)}{dy}-v_j^-(y,q)\frac{dv_{n-j}^+(y,p+q)}{dy}\right.$$

$$\left.+v_j^-(y,p+q)\frac{dv_{n-j}^+(y,q)}{dy}-v_j^+(y,p+q)\frac{dv_{n-j}^-(y,q)}{dy}\right]+$$

$$\frac{1}{2}\sum_{q=1}^{p-1}\left[v_j^+(y,q)\frac{dv_{n-j}^-(y,p-q)}{dy}+v_j^-(y,q)\frac{dv_{n-j}^+(y,p-q)}{dy}\right]\Big\}\sin pz$$

cosine equation

$$-p\omega_j v_n^-(y,p)+ku_j^+(y,0)v_{n-j}^-(y,p)+v_j^+(y,0)\frac{dv_{n-j}^+(y,p)}{dy}+v_j^+(y,p)\frac{dv_{n-j}^+(y,0)}{dy}+$$

$$\frac{1}{2}\sum_{q=1}^{\infty}\Big\{qk\left(u_j^+(y,p+q)v_{n-j}^-(y,q)-u_j^-(y,p+q)v_{n-j}^+(y,q)\right)+$$

$$k(p-q)\left(u_j^+(y,q)v_{n-j}^-(y,p+q)-u_j^-(y,q)v_{n-j}^+(y,p+q)\right)+$$

$$\left[v_j^+(y,p+q)\frac{dv_{n-j}^+(y,q)}{dy}+v_j^-(y,p+q)\frac{dv_{n-j}^-(y,q)}{dy}+\right.$$

$$\left.v_j^+(y,q)\frac{dv_{n-j}^+(y,p+q)}{dy}+v_j^-(y,q)\frac{dv_{n-j}^-(y,p+q)}{dy}\right]\Big\}+$$

$$\frac{1}{2}\sum_{q=1}^{p-1}\Big\{k(p-q)\left[u_j^+(y,q)v_{n-j}^-(y,p-q)+u_j^-(y,q)v_{n-j}^+(y,p-q)\right]+$$

$$\left[v_j^+(y,q)\frac{dv_{n-j}^+(y,p-q)}{dy}-v_j^-(y,q)\frac{dv_{n-j}^-(y,p-q)}{dy}\right]\Big\}$$

sine equation

$$p\omega_j v_n^+(y,p)-ku_j^+(y,0)v_{n-j}^+(y,p)+v_j^+(y,0)\frac{dv_{n-j}^-(y,p)}{dy}+v_j^-(y,p)\frac{dv_{n-j}^+(y,0)}{dy}+$$

$$\frac{1}{2}\sum_{q=1}^{\infty}\left\{qk\left(u_j^-(y,p+q)v_{n-j}^-(y,q)+u_j^+(y,p+q)v_{n-j}^+(y,q)\right)+\right.$$
$$k(p+q)\left(u_j^-(y,q)v_{n-j}^-(y,p+q)+u_j^+(y,q)v_{n-j}^+(y,p+q)\right)+$$
$$\left[v_j^+(y,q)\frac{dv_{n-j}^-(y,p+q)}{dy}-v_j^-(y,q)\frac{dv_{n-j}^+(y,p+q)}{dy}\right.$$
$$\left.\left.+v_j^-(y,p+q)\frac{dv_{n-j}^+(y,q)}{dy}-v_j^+(y,p+q)\frac{dv_{n-j}^-(y,q)}{dy}\right]\right\}+$$
$$\frac{1}{2}\sum_{q=1}^{p-1}\left\{k(p-q)\left(u_j^-(y,q)v_{n-j}^-(y,p-q)-u_j^+(y,q)v_{n-j}^+(y,p-q)\right)+\right.$$
$$\left.\left.v_j^+(y,q)\frac{dv_{n-j}^-(y,p-q)}{dy}+v_j^-(y,q)\frac{dv_{n-j}^+(y,p-q)}{dy}\right]\right\}$$

zero equation

$$v_j^-(y,0)\frac{dv_{n-j}^+(y,0)}{dy}+$$
$$\frac{1}{2}\sum_{j=1}^{\infty}\left\{qk\left[u_j^+(y,q)v_{n-j}^-(y,q)-u_j^-(y,q)v_{n-j}^+(y,q)\right]+\left[v_j^+(y,q)\frac{dv_{n-j}^+(y,q)}{dy}+v_j^-(y,q)\frac{dv_{n-j}^-(y,q}{dy}\right.\right.$$

joint equations

$$=p\omega v_n^\pm(y,p)\pm ku_j^+(y,0)v_{n-j}^\pm(y,p)+v_j^+(y,0)\frac{dv_{n-j}^\pm(y,p)}{dy}+v_j^\pm(y,p)\frac{dv_{n-j}^+(y,0)}{dy}+$$
$$\frac{1}{2}\sum_{j=1}^{\infty}\left\{qk\left(u_j^\pm(y,p+q)v_{n-j}^-(y,q)\mp u_j^+(y,p+q)v_{n-j}^\pm(y,q)\right)\pm\right.$$
$$k(p+q)\left(u_j^\mp(y,q)v_{n-j}^\pm(y,p+q)\mp u_j^\pm(y,q)v_{n-j}^\mp(y,p+q)\right)+$$
$$\pm v_j^-(y,p+q)\frac{dv_{n-j}^+(y,q)}{dy}+v_j^\pm(y,p+q)\frac{dv_{n-j}^+(y,q)}{dy}+$$
$$\left.\left.v_j^\mp(y,q)\frac{dv_{n-j}^\pm(y,p+q)}{dy}\mp v_j^-(y,q)\frac{dv_{n-j}^\mp(y,p+q)}{dy}\right]\right\}$$

THIRD EQUATION

$$-\sum_{j=1}^{n-1}\left\{\omega_j\frac{\partial\Phi_{n-j}}{\partial z}+ku_j\frac{\partial\Phi_{n-j}}{\partial z}+v_j\frac{\partial\Phi_{n-j}}{\partial y}+\Phi_j(k\frac{\partial u_{n-j}}{\partial z}+\frac{\partial v_{n-j}}{\partial y})\right\}$$

First term

$$\omega_j\sum_{p=1}^{\infty}p\left[-\Phi_{n-j}^-(y,p)\cos pz+\Phi_{n-j}^+(y,p)\sin pz\right]$$

Second term

$$\sum_{p=0}^{\infty}\left[u_j^+(y,p)\cos pz+u_j^-(y,p)\sin pz\right]\sum_{p=1}^{\infty}p\left[\Phi_{n-j}^-(y,p)\cos pz-\Phi_{n-j}^+(y,p)\sin pz\right]=$$
$$\frac{1}{2}\sum_{q=1}^{\infty}q\left[u_j^+(y,q)\Phi_{n-j}^-(y,q)-u_j^-(y,q)\Phi_{n-j}^+(y,q)\right]+$$
$$\sum_{p=1}^{\infty}\left\{pu_j^+(y,0)\Phi_{n-j}^-(y,p)+\right.$$
$$\frac{1}{2}\sum_{q=1}^{\infty}\left[q\left(u_j^+(y,p+q)\Phi_{n-j}^-(y,q)-u_j^-(y,p+q)\Phi_{n-j}^+(y,q)\right)+\right.$$

$$(p+q)\left(u_j^+(y,q)\Phi_{n-j}^-(y,p+q) - u_j^-(y,q)\Phi_{n-j}^+(y,p+q)\right)\Big] +$$

$$\frac{1}{2}\sum_{q=1}^{p-1}(p-q)\left[u_j^+(y,q)\Phi_{n-j}^-(y,p-q) + u_j^-(y,q)\Phi_{n-j}^+(y,p-q)\right]\Bigg\}\cos pz +$$

$$\sum_{p=1}^{\infty}\Big\{-u_j^+(y,0)\Phi_{n-j}^+(y,p) +$$

$$\frac{1}{2}\sum_{q=1}^{\infty}\Big[q\left(u_j^-(y,p+q)\Phi_{n-j}^-(y,q) + u_j^+(y,p+q)\Phi_{n-j}^+(y,q)\right) -$$

$$(p+q)\left(u_j^+(y,q)\Phi_{n-j}^+(y,p+q) + u_j^-(y,q)\Phi_{n-j}^-(y,p+q)\right)\Big] +$$

$$\frac{1}{2}\sum_{q=1}^{p-1}(p-q)\left[u_j^-(y,q)\Phi_{n-j}^-(y,p-q) - u_j^+(y,q)\Phi_{n-j}^+(y,p-q)\right]\Bigg\}\sin pz$$

Third term

$$\sum_{j=0}^{\infty}\left[v_j^+(y,p)\cos pz + v_j^-(y,p)\sin pz\right]\sum_{p=0}^{\infty}\left[\frac{d\Phi_{n-j}^+(y,p)}{dy}\cos pz + \frac{d\Phi_{n-j}^-(y,p)}{dy}\sin pz\right] =$$

$$v_j^+(y,0)\frac{d\Phi_{n-j}^+(y,0)}{dy} + \frac{1}{2}\sum_{q=1}^{\infty}\left[v_j^+(y,q)\frac{d\Phi_{n-j}^+(y,q)}{dy} + v_j^-(y,q)\frac{d\Phi_{n-j}^-(y,q)}{dy}\right] +$$

$$\sum_{p=1}^{\infty}\Bigg\{v_j^+(y,0)\frac{d\Phi_{n-j}^+(y,p)}{dy} + v_j^+(y,p)\frac{d\Phi_{n-j}^+(y,0)}{dy} +$$

$$\frac{1}{2}\sum_{q=1}^{\infty}\Big[v_j^+(y,p+q)\frac{d\Phi_{n-j}^+(y,q)}{dy} + v_j^-(y,p+q)\frac{d\Phi_{n-j}^-(y,q)}{dy} +$$

$$v_j^+(y,q)\frac{d\Phi_{n-j}^+(y,p+q)}{dy} + v_j^-(y,q)\frac{d\Phi_{n-j}^-(y,p+q)}{dy}\Big] +$$

$$\frac{1}{2}\sum_{q=1}^{p-1}\left[v_j^+(y,q)\frac{d\Phi_{n-j}^+(y,p-q)}{dy} - v_j^-(y,q)\frac{d\Phi_{n-j}^-(y,p-q)}{dy}\right]\Bigg\}\cos pz +$$

$$\sum_{p=1}^{\infty}\Bigg\{v_j^+(y,0)\frac{d\Phi_{n-j}^-(y,p.)}{dy} + v_j^-(y,p)\frac{d\Phi_{n-j}^+(y,0)}{dy} +$$

$$\frac{1}{2}\sum_{q=1}^{\infty}\Big[v_j^+(y,q)\frac{d\Phi_{n-j}^-(y,p+q)}{dy} - v_j^-(y,q)\frac{d\Phi_{n-j}^+(y,p+q)}{dy} -$$

$$v_j^+(y,p+q)\frac{d\Phi_{n-j}^-(y,q)}{dy} + v_j^-(y,p+q)\frac{d\Phi_{n-j}^+(y,q)}{dy}\Big]$$

$$\frac{1}{2}\sum_{q=1}^{p-1}\left[v_j^+(y,q)\frac{d\Phi_{n-j}^-(y,p-q)}{dy} + v_j^-(y,q)\frac{d\Phi_{n-j}^+(y,p-q)}{dy}\right]\Bigg\}\sin pz$$

fourth term

$$\Phi_j(k\frac{\partial u_{n-j}}{\partial z} + \frac{\partial v_{n-j}}{\partial y})$$

$$\sum_{p=1}^{\infty}pk\left[u_{n-j}^-(y,p)\cos pz - u_{n-j}^+(y,p)\sin pz\right] + \sum_{p=0}^{\infty}\left[\frac{dv_{n-j}^+(y,p)}{dy}\cos pz + \frac{dv_{n-j}^-(y,p)}{dy}\sin pz\right] =$$

$$\frac{dv_{n-j}^+(y,0)}{dy} + \sum_{p=1}^{\infty}\left[\left(pku_{n-j}^-(y,p) + \frac{dv_{n-j}^+(y,p)}{dy}\right)\cos pz - \left(pku_{n-j}^+(y,p) - \frac{dv_{n-j}^-(y,p)}{dy}\right)\sin pz\right]$$

$$\sum_{j=0}^{\infty}\left[\Phi_j^+(y,p)\cos pz + \Phi_j^-(y,p)\sin pz\right] *$$

$$\Bigg\{\frac{dv_{n-j}^+(y,0)}{dy} + \sum_{p=1}^{\infty}\left[\left(pku_{n-j}^-(y,p) + \frac{dv_{n-j}^+(y,p)}{dy}\right)\cos pz - \left(pku_{n-j}^+(y,p) - \frac{dv_{n-j}^-(y,p)}{dy}\right)\sin p\right.$$

$$\Phi_j^+(y,0)\frac{dv_{n-j}^+(y,0)}{dy}+\frac{1}{2}\sum_{q=1}^{\infty}\left[\Phi_j^+(y,q)\left(qku_{n-j}(y,q)+\frac{dv_{n-j}^+(y,q)}{dy}\right)-\right.$$
$$\left.\Phi_j^-(y,q)\left(qku_{n-j}^+(y,q)-\frac{dv_{n-j}^-(y,q)}{dy}\right)\right]+$$
$$\sum_{p=1}^{\infty}\left\{\Phi_j^+(y,0)\left(pku_{n-j}^-(y,p)+\frac{dv_{n-j}^+(y,p)}{dy}\right)+\Phi_j^+(y,p)\frac{dv_{n-j}^+(y,0)}{dy}+\right.$$
$$\frac{1}{2}\sum_{q=1}^{\infty}\left[\Phi_j^+(y,p+q)\left(qku_{n-j}^-(y,q)+\frac{dv_{n-j}^+(y,q)}{dy}\right)-\Phi_j^-(y,p+q)\left(qku_{n-j}^+(y,q)-\frac{dv_{n-j}^-(y,q)}{dy}\right)\right.$$
$$\Phi_j^+(y,q)\left((p+q)\,ku_{n-j}^-(y,p+q)+\frac{dv_{n-j}^+(y,p+q)}{dy}\right)-$$
$$\left.\Phi_j^-(y,q)\left((p+q)\,ku_{n-j}^+(y,p+q)-\frac{dv_{n-j}^-(y,p+q)}{dy}\right)\right]+$$
$$\frac{1}{2}\sum_{q=1}^{p-1}\left[\Phi_j^+(y,q)\left((p-q)\,ku_{n-j}^-(y,p-q)+\frac{dv_{n-j}^+(y,p-q)}{dy}\right)+\right.$$
$$\left.\left.\Phi_j^-(y,q)\left((p-q)\,ku_{n-j}^+(y,p-q)-\frac{dv_{n-j}^-(y,p-q)}{dy}\right)\right]\right\}\cos pz+$$
$$\sum_{p=1}^{\infty}\left\{-\Phi_j^-(y,0)\left(pku_{n-j}^+(y,p)-\frac{dv_{n-j}^-(y,p)}{dy}\right)+\Phi_j^-(y,p)\frac{dv_{n-j}^+(y,0)}{dy}+\right.$$
$$\frac{1}{2}\sum_{q=1}^{\infty}\left[-\Phi_j^-(y,q)\left((p+q)\,ku_{n-j}^+(y,p+q)-\frac{dv_{n-j}^-(y,p+q)}{dy}\right)-\right.$$
$$\Phi_j^+(y,q)\left((p+q)\,ku_{n-j}^-(y,p+q)+\frac{dv_{n-j}^+(y,p+q)}{dy}\right)+$$
$$\Phi_j^-(y,p+q)\left(qku_{n-j}^+(y,q)-\frac{dv_{n-j}^-(y,q)}{dy}\right)+$$
$$\left.\Phi_j^-(y,p+q)\left(qku_{n-j}^-(y,q)+\frac{dv_{n-j}^+(y,q)}{dy}\right)\right]+$$
$$\frac{1}{2}\sum_{q=1}^{p-1}\left[-\Phi_j^-(y,q)\left((p-q)\,ku_{n-j}^+(y,p-q)-\frac{dv_{n-j}^-(y,p-q)}{dy}\right)+\right.$$
$$\left.\left.\Phi_j^+(y,q)\left((p-q)\,ku_{n-j}(y,p-q)+\frac{dv_{n-j}^+(y,p-q)}{dy}\right)\right]\right\}\sin pz$$

cosine equation

$$-p\omega_j\Phi_{n-j}^-(y,p)+pku_j^+(y,0)\Phi_{n-j}^-(y,p)+v_j^+(y,0)\frac{d\Phi_{n-j}^+(y,p)}{dy}+v_j^+(y,p)\frac{d\Phi_{n-j}^+(y,0)}{dy}+$$
$$\Phi_j^+(y,0)\left(pku_{n-j}(y,p)+\frac{dv_{n-j}^+(y,p)}{dy}\right)+\Phi_j^+(y,p)\frac{dv_{n-j}^+(y,0)}{dy}$$
$$+\frac{1}{2}\sum_{q=1}^{\infty}\left\{\left[qk\left(u_j^+(y,p+q)\Phi_{n-j}^-(y,q)-u_j^-(y,p+q)\Phi_{n-j}^+(y,q)\right)+\right.\right.$$
$$\left.k\,(p+q)\left(u_j^+(y,q)\Phi_{n-j}^-(y,p+q)-u_j^-(y,q)\Phi_{n-j}^+(y,p+q)\right)\right]$$
$$+\left[v_j^+(y,p+q)\frac{d\Phi_{n-j}^+(y,q)}{dy}+v_j^-(y,p+q)\frac{d\Phi_{n-j}^-(y,q)}{dy}+\right.$$

$$v_j^+(y,q)\frac{d\Phi^+_{n-j}(y,p+q)}{dy} + v_j^-(y,q)\frac{d\Phi^-_{n-j}(y,p+q)}{dy}\Big]$$

$$+\Big[\Phi_j^+(y,p+q)\left(qku_{n-j}^-(y,q)+\frac{dv^+_{n-j}(y,q)}{dy}\right) - \Phi_j^-(y,p+q)\left(qku^+_{n-j}(y,q) - \frac{dv^-_{n-j}(y,q)}{dy}\right)+$$

$$\Phi_j^+(y,q)\left((p+q)ku^-_{n-j}(y,p+q)+\frac{dv^+_{n-j}(y,p+q)}{dy}\right) -$$

$$\Phi_j^-(y,q)\left((p+q)ku^+_{n-j}(y,p+q) - \frac{dv^-_{n-j}(y,p+q)}{dy}\right)\Big]\Big\}$$

$$-\frac{1}{2}\sum_{q=1}^{p-1}\Big\{k(p-q)\Big[u_j^+(y,q)\Phi^-_{n-j}(y,p-q)+u_j^-(y,q)\Phi^+_{n-j}(y,p-q)\Big]$$

$$-\Big[v_j^+(y,q)\frac{d\Phi^+_{n-j}(y,p-q)}{dy} - v_j^-(y,q)\frac{d\Phi^-_{n-j}(y,p-q)}{dy}\Big]$$

$$-\Big[\Phi_j^+(y,q)\left((p-q)ku^-_{n-j}(y,p-q)+\frac{dv^+_{n-j}(y,p-q)}{dy}\right)+$$

$$\Phi_j^-(y,q)\left((p-q)ku^+_{n-j}(y,p-q)-\frac{dv^-_{n-j}(y,p-q)}{dy}\right)\Big]\Big\}$$

sine equation

$$\omega\Phi^-_n(y,p) = ku_j^+(y,0)\Phi^+_{n-j}(y,p) + v_j^+(y,0)\frac{d\Phi^-_{n-j}(y,p)}{dy} + v_j^-(y,p)\frac{d\Phi^+_{n-j}(y,0)}{dy}$$

$$-\Phi_j^+(y,0)\left(pku^+_{n-j}(y,p) - \frac{dv^-_{n-j}(y,p)}{dy}\right) + \Phi_j^-(y,p)\frac{dv^+_{n-j}(y,0)}{dy}$$

$$-\frac{1}{2}\sum_{q=1}^{\infty}\Big\{\Big[qk\left(u_j^-(y,p+q)\Phi^-_{n-j}(y,q)+u_j^+(y,p+q)\Phi^+_{n-j}(y,q)\right) -$$

$$(p+q)k\left(u_j^+(y,q)\Phi^+_{n-j}(y,p+q)+u_j^-(y,q)\Phi^-_{n-j}(y,p+q)\right)\Big]$$

$$-\Big[v_j^+(y,q)\frac{d\Phi^-_{n-j}(y,p+q)}{dy} - v_j^-(y,q)\frac{d\Phi^+_{n-j}(y,p+q)}{dy}$$

$$-v_j^+(y,p+q)\frac{d\Phi^-_{n-j}(y,q)}{dy} + v_j^-(y,p+q)\frac{d\Phi^+_{n-j}(y,q)}{dy}\Big]$$

$$-\Big[-\Phi_j^+(y,q)\left((p+q)ku^+_{n-j}(y,p+q)-\frac{dv^-_{n-j}(y,p+q)}{dy}\right)+$$

$$\Phi_j^-(y,q)\left((p+q)ku^-_{n-j}(y,p+q)+\frac{dv^+_{n-j}(y,p+q)}{dy}\right)+$$

$$\Phi_j^+(y,p+q)\left(qku^+_{n-j}(y,q)-\frac{dv^-_{n-j}(y,q)}{dy}\right)+$$

$$\Phi_j^-(y,p+q)\left(qku^-_{n-j}(y,q)+\frac{dv^+_{n-j}(y,q)}{dy}\right)\Big]\Big\}$$

$$+\frac{1}{2}\sum_{q=1}^{p-1}\Big\{k(p-q)\Big[u_j^-(y,q)\Phi^+_{n-j}(y,p-q) - u_j^+(y,q)\Phi^-_{n-j}(y,p-q)\Big]$$

$$+\Big[v_j^+(y,q)\frac{d\Phi^-_{n-j}(y,p-q)}{dy}+v_j^-(y,q)\frac{d\Phi^+_{n-j}(y,p-q)}{dy}\Big]$$

$$+\Big[-\Phi_j^+(y,q)\left((p-q)ku^+_{n-j}(y,p-q)-\frac{dv^-_{n-j}(y,p-q)}{dy}\right)+$$

$$\Phi_j^-(y,q)\left((p-q)\,ku_{n-j}^-(y,p-q)+\frac{dv_{n-j}^+(y,p-q)}{dy}\right)\Bigg]\Bigg\}$$

zero equation

$$v_j^+(y,0)\frac{d\Phi_{n-j}^+(y,0)}{dy}$$
$$-\frac{1}{2}\sum_{q=1}^{\infty}\left\{qk\left[u_j^+(y,q)\Phi_{n-j}^-(y,q)-u_j^-(y,q)\Phi_{n-j}^+(y,q)\right]\right.$$
$$-\left[v_j^+(y,q)\frac{d\Phi_{n-j}^+(y,q)}{dy}+v_j^-(y,q)\frac{d\Phi_{n-j}^-(y,q)}{dy}\right]+$$
$$\Phi_j^+(y,0)\frac{dv_{n-j}^+(y,0)}{dy}+\left[\Phi_j^+(y,q)\left(qku_{n-j}^-(y,q)+\frac{dv_{n-j}^+(y,q)}{dy}\right)-\right.$$
$$\left.\left.\Phi_j^-(y,q)\left(qku_{n-j}^+(y,q)-\frac{dv_{n-j}^-(y,q)}{dy}\right)\right]\right\}$$

joint equations

$$\mp p\omega_j\Phi_{n-j}^{\mp}(y,p)\pm pku_j^+(y,0)\Phi_{n-j}^{\mp}(y,p)+v_j^+(y,0)\frac{d\Phi_{n-j}^{\pm}(y,p)}{dy}+v_j^{\pm}(y,p)\frac{d\Phi_{n-j}^+(y,0)}{dy}\pm$$
$$\Phi_j^+(y,0)\left(pku_{n-j}^{\mp}(y,p)\pm\frac{dv_{n-j}^{\pm}(y,p)}{dy}\right)+\Phi_j^{\pm}(y,p)\frac{dv_{n-j}^+(y,0)}{dy}$$
$$-\frac{1}{2}\sum_{q=1}^{\infty}\left\{\left[qk\left(u_j^{\pm}(y,p+q)\Phi_{n-j}^-(y,q)\pm u_j^{\mp}(y,p+q)\Phi_{n-j}^+(y,q)\right)\pm\right.\right.$$
$$\left.(p+q)\left(u_j^+(y,q)\Phi_{n-j}^{\mp}(y,p+q)\mp u_j^-(y,q)\Phi_{n-j}^{\pm}(y,p+q)\right)\right]$$
$$-\left[\pm v_j^{\mp}(y,p+q)\frac{d\Phi_{n-j}^{\pm}(y,q)}{dy}\mp v_j^{\pm}(y,p+q)\frac{d\Phi_{n-j}^{\mp}(y,q)}{dy}+\right.$$
$$\left.v_j^+(y,q)\frac{d\Phi_{n-j}^{\pm}(y,p+q)}{dy}\pm v_j^-(y,q)\frac{d\Phi_{n-j}^{\mp}(y,p+q)}{dy}\right]$$
$$-\left[\Phi_j^{\mp}(y,p+q)\left(qku_{n-j}^{\mp}(y,q)\pm\frac{dv_{n-j}^{\pm}(y,q)}{dy}\right)\mp\Phi_j^{\pm}(y,p+q)\left(qku_{n-j}^{\pm}(y,q)\mp\frac{dv_{n-j}^{\mp}(y,q)}{dy}\right)\pm\right.$$
$$\Phi_j^+(y,q)\left((p+q)\,ku_{n-j}^{\mp}(y,p+q)\pm\frac{dv_{n-j}^{\pm}(y,p+q)}{dy}\right)-$$
$$\left.\left.\Phi_j^-(y,q)\left((p+q)\,ku_{n-j}^{\pm}(y,p+q)\mp\frac{dv_{n-j}^{\mp}(y,p+q)}{dy}\right)\right]\right\}$$