Solution of the inverse problem in spherical gravitational wave detectors using a model with independent bars

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The direct detection of gravitational waves will provide valuable astrophysical information about many celestial objects. The SCHENBERG has already undergone its first test run. It is expected to have its first scientific run soon. In this work a new data analysis approach is presented, called the method of independent bars, which can be used with SCHENBERG's data. We test this method through the simulation of the detection of gravitational waves. With this method we find the source's direction without the need to have all six transducers operational. Also, we show that the method is a generalization of another one, already described in the literature, known as the mode channels method.

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I. INTRODUCTION

The detection of gravitational waves will have many implications in physics and astrophysics. Besides the confirmation of general relativity theory, it will allow the investigation of several astrophysical phenomena, such as the existence of black holes and the mass and abundance of neutron stars, thus allowing new scientific frontiers.

Joseph Weber was the first to propose feasible gravitational wave detectors in the 1960's [1]. The antenna he proposed had a cylindrical shape, operated at room temperature, and was acoustically isolated. Since his pioneering work the detectors based on resonant-mass antennas have improved significantly. For the latest generation of such detectors, the antenna has a spherical shape.

The transducer distribution on spherical detectors follows the truncated icosahedron configuration, proposed by Johnson and Merkowitz in 1993 [2] and justified formally by Magalhães *et al.* in 1997 [3]. Spherical antennas have omnidirectional sensitivity, and all observables needed for gravitational astronomy can be obtained from only one spherical detector appropriately equipped with six transducers [4]. When fully operational, such a detector will be able not only to acknowledge the presence of a gravitational wave within its bandwidth, but also to detect the direction of a source in the sky as well as the amplitudes of the wave's polarization components. This is the case of SCHENBERG, the second spherical detector ever built in the world and the first equipped with a set of parametric transducers; it is installed at the Physics Institute of the University of Sao Paulo (Sao Paulo, Brazil). It underwent its first test run in September 8, 2006, with three transducers operational. Recent information on the present status of this detector can be found in Ref. [5].

There are mathematical models for this detector for the case that all six transducers are operational. Such models investigated two situations: one in which the transducers were perfectly uncoupled [3,6,7] and another in which the transducers were somehow coupled to each other [8]. In both cases it is possible to solve the inverse problem and retrieve the four astrophysical parameters of interest: the angles that determine the source's position in the sky, denoted by β (zenith angle) and γ (azimuthal angle); and the amplitudes of the two polarization states as functions of time, $h_+(t)$ and $h_\times(t)$.

In this work we present a methodology for the solution of the inverse problem in spherical detectors, which we named "the method of independent bars." Unlike the previous methods, it allows the determination of astrophysical parameters using less than six transducers. In fact, we will show that it generalizes a previous approach, known as the mode channels method. We tested our method with simulations and found encouraging results. In the next sections we will show these results in detail.

II. THE METHOD OF INDEPENDENT BARS

The determination of directions in the sky demands the use of appropriate reference frames. For this reason we shall define two frames: the first one is the *laboratory system* with axes defined by the unit vectors $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$. The second frame is the *gravitational wave system*, which will have its axes defined by $(\mathbf{e}_X, \mathbf{e}_Y, \mathbf{e}_Z)$ with the wave traveling in the $\mathbf{e}_Z$ direction and the axes of the polarization amplitudes being $(\mathbf{e}_X, \mathbf{e}_Y)$.

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The projection of the polarization state amplitudes of the gravitational signal on the shaft of a detector can be expressed by the contraction of two symmetric, trace-free tensors [3]:

$$R(t) = \tfrac{1}{2}h^{ij}(t)D_{ij}. \tag{1}$$

The response amplitude of one transducer coupled to the spherical detector is given by $R(t)$, while $h^{ij}(t)$ is the spatial part of the gravitational wave tensor and D_{ij} is the transducer's tensor, which gives the orientation of the transducer on the spherical detector.

In Appendix B we show that the contraction in Eq. (1) is a representation of the time-dependent scalar potential given by [6]

$$\phi(\mathbf{x}, t) = \tfrac{1}{2}\rho x_j \ddot{h}^{jk}(t)x_k, \tag{2}$$

where ρ is the antenna's mass density and x_i is a coordinate location.

A. The transducer's tensor

It is possible to model the system as if we had one independent bar behind each transducer on the detector [4]. Such a bar detector, whose axis is in the $\mathbf{n}$ direction, can be characterized by the symmetric trace-free tensor [9]

$$D_{ij} = (\mathbf{n} \otimes \mathbf{n})_{ij} - \tfrac{1}{3}\delta_{ij}. \tag{3}$$

The term $\frac{1}{3}\delta_{ij}$ ensures the property of tracelessness of the tensor.

The transducers are located on the antenna, and the position of the Lth transducer is described by the unit vector $\mathbf{n}$. In the laboratory system (lab frame) this vector is given by

$$\mathbf{n}_L = \sin\theta_L \cos\phi_L \mathbf{e}_x + \sin\theta_L \sin\phi_L \mathbf{e}_y + \cos\theta_L \mathbf{e}_z, \tag{4}$$

where θ_L and ϕ_L are angles that locate the transducer on the spherical detector. Using Eqs. (3) and (4) the Lth transducer's tensor assumes the following form in the lab frame:

$$\mathbf{D}^L = \begin{bmatrix} \sin^2\theta_L\cos^2\phi_L - \frac{1}{3} & \frac{1}{2}\sin^2\theta_L \sin2\phi_L & \frac{1}{2}\sin2\theta_L \cos\phi_L \\ \frac{1}{2}\sin^2\theta_L \sin2\phi_L & \sin^2\theta_L\sin^2\phi_L - \frac{1}{3} & \frac{1}{2}\sin2\theta_L \sin\phi_L \\ \frac{1}{2}\sin2\theta_L \cos\phi_L & \frac{1}{2}\sin2\theta_L \sin\phi_L & \cos^2\theta_L - \frac{1}{3} \end{bmatrix}. \tag{5}$$

B. The wave's tensor

Consider the complex null vector $\mathbf{m}$ defined by [10]

$$\mathbf{m} = \frac{1}{\sqrt{2}}(\mathbf{e}_X + i\mathbf{e}_Y), \tag{6}$$

where $\mathbf{e}_X$ and $\mathbf{e}_Y$ are unit vectors in the wave frame. The gravitational wave tensor $\mathbf{h}$ can be written in terms of these vectors:

$$\mathbf{h} = 2h_+\mathrm{Re}(\mathbf{m} \otimes \mathbf{m}) + 2h_\times\mathrm{Im}(\mathbf{m} \otimes \mathbf{m}) \tag{7}$$

or

$$\mathbf{h} = h_+\mathbf{e}_+ + h_\times\mathbf{e}_\times, \tag{8}$$

where

$$\mathbf{e}_+ = \mathbf{e}_X \otimes \mathbf{e}_X - \mathbf{e}_Y \otimes \mathbf{e}_Y,$$

$$\mathbf{e}_\times = \mathbf{e}_X \otimes \mathbf{e}_Y + \mathbf{e}_Y \otimes \mathbf{e}_X.$$

Starting from the lab frame $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ it is possible to rotate by γ around $\mathbf{e}_z$ to obtain $(\mathbf{e}_{x'}, \mathbf{e}_{y'}, \mathbf{e}_{z'})$ and rotate by β around $\mathbf{e}_{y'}$ (Euler y convention) to obtain $(\mathbf{e}_{x''}, \mathbf{e}_{y''}, \mathbf{e}_{z''})$ so that $\mathbf{e}_{z''}$ points in the direction of the propagation of the wave $\mathbf{e}_Z$.

In the lab frame the vector $\mathbf{m}$ thus takes the following form:

$$\mathbf{m} = \frac{1}{\sqrt{2}}[(\cos\beta\cos\gamma - i\sin\gamma)\mathbf{e}_x + (\cos\beta\sin\gamma + i\cos\gamma)\mathbf{e}_y - \sin\beta\mathbf{e}_z], \tag{9}$$

where β, γ are the zenithal and azimuthal angles, respectively, that define the direction of arrival of the gravitational signal.

We have determined the expressions for the wave's and the detector's tensors relative to the laboratory system. We can now solve the inverse problem.

C. The inverse problem

Equation (1), valid for one transducer, can be expanded as [11]

$$\tfrac{1}{2}[D^L_{11}h_{11} + 2D^L_{12}h_{12} + 2D^L_{13}h_{13} + D^L_{22}h_{22} + 2D^L_{23}h_{23} + 2D^L_{33}h_{33}] = R^L. \tag{10}$$

In the case that we have six operating transducers, using the six independent components of the spatial part of the gravitational-waves tensor, the h_{ij} can be obtained as the solutions of the system of equations

$$\begin{bmatrix} D^1_{11} & 2D^1_{12} & 2D^1_{13} & D^1_{22} & 2D^1_{23} & D^1_{33} \\ D^2_{11} & 2D^2_{12} & 2D^2_{13} & D^2_{22} & 2D^2_{23} & D^2_{33} \\ D^3_{11} & 2D^3_{12} & 2D^3_{13} & D^3_{22} & 2D^3_{23} & D^3_{33} \\ D^4_{11} & 2D^4_{12} & 2D^4_{13} & D^4_{22} & 2D^4_{23} & D^4_{33} \\ D^5_{11} & 2D^5_{12} & 2D^5_{13} & D^5_{22} & 2D^5_{23} & D^5_{33} \\ D^6_{11} & 2D^6_{12} & 2D^6_{13} & D^6_{22} & 2D^6_{23} & D^6_{33} \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{22} \\ h_{23} \\ h_{33} \end{bmatrix} = 2\begin{bmatrix} R^1 \\ R^2 \\ R^3 \\ R^4 \\ R^5 \\ R^6 \end{bmatrix}. \tag{11}$$

On the other hand, in the case where the detector is coupled to a set of only five transducers ($L = 1, \ldots, 5$), the last line will be absent. So in its place we use the trace-free condition [11]

$$h_{11} + h_{22} + h_{33} = 0. \tag{12}$$

When only four transducers are operational, it is still possible to solve the system of equations by adding to it the transversality condition of the gravitational wave. This condition implies that the spatial part of the gravitational wave tensor is orthogonal to the direction of propagation, $h_{ij}n^j = 0$; i.e., the matrix h_{ij} has one eigenvalue equal to zero and, as a consequence, a null determinant. We then use the condition of the null determinant, given by

$$h_{11}h_{22}h_{33} + 2h_{12}h_{13}h_{23} - h_{33}h_{12}^2 - h_{22}h_{13}^2 - h_{11}h_{23}^2 = 0, \tag{13}$$

instead of the line that is missing in the system of equations (11).

Since we take general relativity for granted, the h matrix, without noise, must be "TT": transverse [Eq. (13)] and trace free, [Eq. (12)]. In this case any set of 6 $h_{\mu\nu}$, obtained as described above, should obey such conditions. The presence of noise is expected to make the $h_{\mu\nu}$ vary so that the transversality and tracelessness of $\mathbf{h}$ are not perfect. Of course, if the noise is too high it will be difficult to identify the TT properties in the $\mathbf{h}$ matrix. Our ability to extract the signal from the noise will depend on the signal-to-noise ratio (SNR) achieved for the particular event. This has been investigated by Merkowitz [12] and Merkowitz, Lobo and Serrano [13], partly using an approach first presented in Magalhães *et al.* [14]. In the present work we will not approach this issue in depth, but leave it to be discussed elsewhere.

III. THE ASTROPHYSICAL PARAMETERS

With the $\mathbf{h}$ obtained from the system of equations (11) it is possible to calculate all the relevant parameters of the gravitational wave: the angles β and γ that indicate the direction of the wave, and the amplitudes of the two polarization states of the wave as functions of time, $h_+(t)$ and $h_\times(t)$.

A. The direction of the gravitational wave

For a gravitational wave that propagates in the direction of the $\mathbf{e}_Z$ axis of its proper system of coordinates, in the case of general relativity the transversality condition relation holds [9,15]:

$$\mathbf{h} \cdot \mathbf{e}_Z = 0. \tag{14}$$

The wave frame can be written in terms of the laboratory frame through a rotation. Following the y convention for the Euler angles, $\mathbf{e}_Z$ assumes the form

$$\mathbf{e}_Z = \sin\beta\cos\gamma\mathbf{e}_x + \sin\beta\sin\gamma\mathbf{e}_y + \cos\beta\mathbf{e}_z. \tag{15}$$

Substituting Eq. (15) in (14) and using the symmetry of h_{ij}, we obtain the following system of equations:

$$h_{11}\sin\beta\cos\gamma + h_{12}\sin\beta\sin\gamma + h_{13}\cos\gamma = 0, \tag{16a}$$

$$h_{12}\sin\beta\cos\gamma + h_{22}\sin\beta\sin\gamma + h_{23}\cos\gamma = 0, \tag{16b}$$

$$h_{13}\sin\beta\cos\gamma + h_{23}\sin\beta\sin\gamma + h_{33}\cos\gamma = 0. \tag{16c}$$

Since $\det\mathbf{h} = 0$, in this system the equations are linearly dependent. Two of them are used to obtain β and γ:

$$\tan\gamma = \frac{h_{12}h_{13} - h_{23}h_{11}}{h_{12}h_{23} - h_{13}h_{22}}, \tag{17}$$

$$\tan\beta = \pm\frac{h_{12}h_{23} - h_{13}h_{22}}{h_{11}h_{22} - h_{12}^2}\sqrt{1 + \tan^2\gamma}. \tag{18}$$

These two solutions for β are related to the two diametrically opposed positions in the sky from which the gravitational wave can arrive.

With the above equations we compute the direction of the incoming wave. It results in the same value for each time sample, in the case of a pure (noiseless) signal. In the case of a noisy signal, a distribution around the mean values of the β and γ angles is found, as we can see in Sec. IV.

B. Amplitudes of the polarization states

Using the fact that $\mathbf{e}_+ \cdot \mathbf{e}_+ = 2$, $\mathbf{e}_\times \cdot \mathbf{e}_\times = 2$, and $\mathbf{e}_+ \cdot \mathbf{e}_\times = 0$, we obtain h_+ and $h_\times$ with the aid of Eq. (8), yielding

$$\begin{aligned} h_+ = {} & (\cos^2\beta\cos^2\gamma - \sin^2\gamma)h_{11} + \sin2\gamma(1 + \cos^2\beta)h_{12} \\ & - \sin2\beta\cos\gamma h_{13} + (\cos^2\beta\sin^2\gamma - \cos^2\gamma)h_{22} \\ & - \sin2\beta\sin\gamma h_{23} + \sin^2\beta h_{33}, \\ h_\times = {} & -\cos\beta\sin2\gamma h_{11} + 2\cos\beta\cos2\gamma h_{12} \\ & + 2\sin\beta\sin\gamma h_{13} + \cos\beta\sin2\gamma h_{22} \\ & - 2\sin\beta\cos\gamma h_{23}. \end{aligned}$$

Once these amplitudes are known, other quantities can be calculated. One of them is the polarization angle Ψ through the relation

$$\tan(2\psi(t)) = \frac{h_x(t)}{h_+(t)}.$$

However, this parameter is actually not relevant for gravitational wave detection purposes since it is just an arbitrary angle between the wave x axis and an axis where the wave has only the instantaneous plus polarization, h_+.

On the other hand, important information can be obtained from the difference between the polarization phases. This can be determined if we analyze the time evolution of the instantaneous polarizations registered in each sample (the polarizations are time dependent). Our ability to determine this phase difference depends on the signal-to-noise ratio for each particular event sampled. Such time varying analysis is beyond the scope of the present work.

IV. TESTING THE METHOD OF INDEPENDENT BARS

With the objective of testing the efficiency of the method, we produced simulated data of the SCHENBERG detector. For the waveform we chose a template from the Laboratory for High Energy Astrophysics (LHEA) of the NASA/GFSC [16], one of the three waveforms used by Duez and collaborators [17,18] and depicted in Fig. 1.

This signal was introduced in a simulator of SCHENBERG [19] that returns the response of the six two-mode transducers positioned on the detector, their positions obeying the truncated icosahedral configuration [20]. The simulation of the detector is possible because its mathematical model is known as well as the mathematical models of its noise sources [21].

Table I shows the angles relative to the position of each transducer on the spherical detector. We simulated the transducers' responses when the detector was excited by a wave of the kind shown in Fig. 1 arriving at the detector in the direction defined by $\beta = 30°$ and $\gamma = 75°$.

In the SCHENBERG simulator the amplitude SNR was estimated by Thorne as [22]

$$\mathrm{SNR} = \sqrt{\int_{-\infty}^{\infty} \tilde{W}(f) \frac{|\tilde{h}^{\mathrm{GW}}(f)|^2}{S_N(f)} df}, \qquad (19)$$

where $\tilde{W}(f)$ represents the applied filter function (chosen, in the present work, as the function of a pass-band filter). $S_N(f)$ is the equivalent noise profile (Gaussian, in this case); it can be obtained from the detector's output when no useful signal is present, representing the total noise energy in the sphere.

The signal-to-noise ratio used was SNR ~ 8, corresponding to a source with behavior as in Fig. 1, located approximately 97 kpc away. It is from this source model that $\tilde{h}^{\mathrm{GW}}(f)$ can be obtained.

We used the output of 5 or 6 transducers to test our model, as follows.

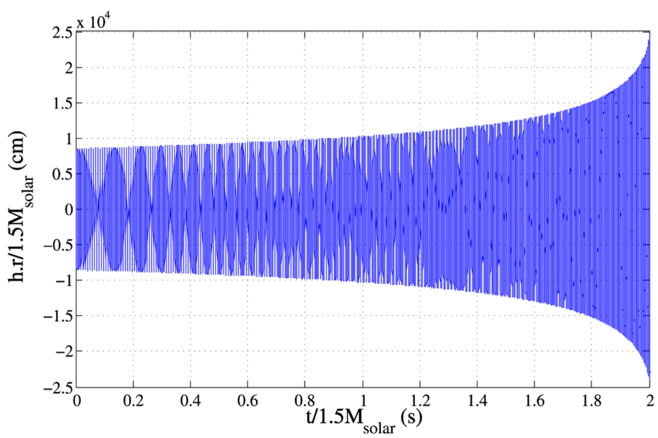

FIG. 1 (color online). Waveform of a gravitational wave emitted by a binary neutron star–neutron star system near its last stable orbits. The time and amplitude axes are scaled according to the object masses through the multiplication by $M/1.5M_{\mathrm{solar}}$, where M_{solar} is the Sun's mass.

TABLE I. Angular position of each one of the transducers on the surface of the Mario SCHENBERG detector.

L	1	2	3	4	5	6
θ_L	79,1877°	79,1877°	79,1877°	37,3774°	37,3774°	37,3774°
ϕ_L	120°	240°	0°	300°	180°	60°

A. Case 1: Pure signal

Once we had the simulated response of the transducers, we calculated the parameters of the gravitational wave using the method of independent bars.

The result obtained with the method for the source's direction was exactly the same as initially imposed: $\beta = 30°$ and $\gamma = 75°$. This shows that the method has good accuracy and does not present intrinsic errors.

B. Case 2: Signal plus noise

We then tested the method in the presence of noise, adding Gaussian noise to the data. To illustrate the results we projected the sphere onto a plane using the Hammer-Aitoff projection [23], where the direction is given by $\theta = 90° - \beta$ and $\phi = -\gamma$.

Figures 2 and 3 show the source's direction as found using the method of independent bars for the cases where the data of six or five transducers are used. In those figures the two regions of the sky are marked because there is a natural ambiguity in the determination of the direction by only one detector. If more than one detector is used, then there is a time delay between detections and one can determine whether the wave came from above or below.

As one can verify, the smaller the number of transducers' data used in the analysis the smaller the precision

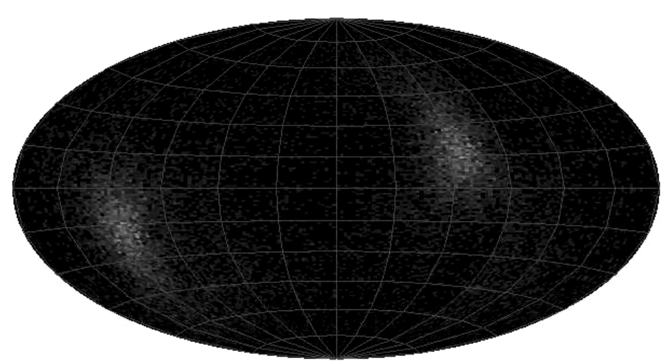

FIG. 2 (color online). Determination of the wave's direction using the method of independent bars in the presence of Gaussian noise with SNR ~ 8. The transducers used are 1, 2, 3, 4, and 6 in Table I.

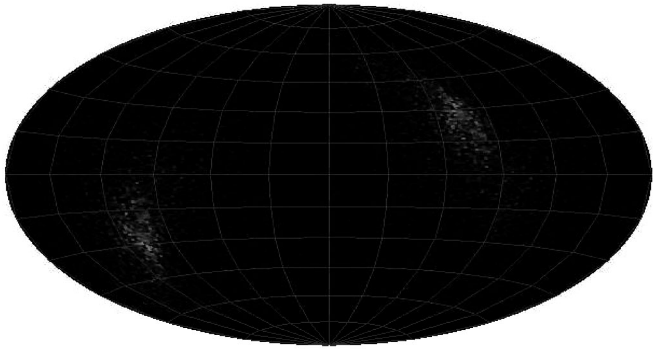

FIG. 3. Determination of the wave's direction using the method of independent bars in the presence of Gaussian noise with SNR ~ 8. The six transducers are positioned according to the truncated icosahedron configuration shown in Table I.

of the results. In those figures we observe that, for the case in which the data of only five transducers are used, the area of interest is larger than in the case where the data of all six transducers are used. This area is equivalent to the error. The numerical results obtained for the method are presented in Table II.

The values presented in this table were obtained simulating the wave's incidence 50 times and then calculating the average and the standard deviation of the results.

We have verified the efficiency of the method of independent bars in determining a source's direction. Another important result is to show the equivalence between the method of independent bars and the mode channels model [6]. This will be done in the next section.

V. EQUIVALENCE BETWEEN THE METHOD OF INDEPENDENT BARS AND THE MODE CHANNELS MODEL

The bar detector response is given in (11). If six transducers are coupled to the detector, we can write the matricial system (12).

Each component of the matrix is a function of the angles θ_L and ϕ_L that gives the position of each transducer. This function can be written as a combination of real spherical harmonics of second order. Therefore the matricial system can be written as

TABLE II. Numerical results obtained with the method of independent bars for the wave's direction in the case of signal in the presence of noise (SNR ~ 8) using 5 or 6 transducers' data. n is the number of transducers.

n	β	γ
5	$(27.2 \pm 7.1)^\circ$	$(78.7 \pm 5.3)^\circ$
6	$(28.1 \pm 4.2)^\circ$	$(75.3 \pm 2.9)^\circ$

$$\begin{bmatrix} R^1 \\ R^2 \\ R^3 \\ R^4 \\ R^5 \\ R^6 \end{bmatrix} = \frac{1}{2}\sqrt{\frac{16\pi}{15}} \begin{bmatrix} Y_1^1 & Y_2^1 & Y_3^1 & Y_4^1 & Y_5^1 \\ Y_1^2 & Y_2^2 & Y_3^2 & Y_4^2 & Y_5^2 \\ Y_1^3 & Y_2^3 & Y_2^3 & Y_4^3 & Y_5^3 \\ Y_1^4 & Y_2^4 & Y_3^4 & Y_4^4 & Y_5^4 \\ Y_1^5 & Y_2^5 & Y_3^5 & Y_4^5 & Y_5^5 \\ Y_1^6 & Y_2^6 & Y_3^6 & Y_4^6 & Y_5^6 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \end{bmatrix}, \tag{20}$$

where h_m are the quadrupolar representation of the gravitational waves. In Appendix A we show the details of this calculation.

When the transducers are positioned according to the truncated icosahedral configuration, the matrix of the real spherical harmonics $\mathbf{Y}$ is the transpose of the model matrix $\mathbf{B}_{6\times5}^T$ proposed by Merkowitz and Johnson [24]. In compact notation Eq. (20) can be written as

$$\mathbf{R} = \frac{1}{2}\sqrt{\frac{16\pi}{15}} \mathbf{B}_{6\times5}^T \mathbf{h}. \tag{21}$$

Using a property of the model matrix [6],

$$\mathbf{B}_{5\times6} \mathbf{B}_{6\times5}^T = \frac{3}{2\pi} \mathbf{I}, \tag{22}$$

we obtain

$$\mathbf{h} = \sqrt{\frac{5\pi}{3}} \mathbf{B}_{5\times6} \mathbf{R}, \tag{23}$$

or, in explicit form,

$$\begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \end{bmatrix} = \sqrt{\frac{5\pi}{3}} \begin{bmatrix} Y_1^1 & Y_1^2 & Y_1^3 & Y_1^4 & Y_1^5 & Y_1^6 \\ Y_2^1 & Y_2^2 & Y_2^3 & Y_2^4 & Y_2^5 & Y_2^6 \\ Y_3^1 & Y_3^2 & Y_3^3 & Y_3^4 & Y_3^5 & Y_3^6 \\ Y_4^1 & Y_4^2 & Y_4^3 & Y_4^4 & Y_4^5 & Y_4^6 \\ Y_5^1 & Y_5^2 & Y_5^3 & Y_5^4 & Y_5^5 & Y_5^6 \end{bmatrix} \begin{bmatrix} R^1 \\ R^2 \\ R^3 \\ R^4 \\ R^5 \\ R^6 \end{bmatrix}. \tag{24}$$

This result, obtained with the independent bars method, coincides with the one found with the mode channels model. Therefore, if the transducers are in a truncated icosahedron configuration, the two models are equivalent.

On the other hand, it is worth stressing that even if the transducers are not in this special configuration, the inverse problem can be solved using the independent bars model.

VI. SUMMARY

The main objective of this work was to show the details of the analytical construction of the method of independent bars and its efficiency in the determination of the relevant parameters of gravitational waves detected by a spherical detector, namely, the two angles that determine the source's position in the sky and the amplitudes of the two polarization states as functions of time.

As can be seen from the results, the method of independent bars is efficient for the solution of the inverse problem. First we tested the method in the noiseless case, and we verified that it does not present intrinsic errors, thus being an exact method. Then we tested the method with Gaussian noise added to the data, as can be seen in Figs. 2 and 3.

A major advantage of the method of independent bars is its generality, since it does not depend on the specific transducer's configuration. The fact that **D** is a square matrix facilitates the inversion of the system's equation, the only requirement being that the matrix is nonsingular.

Therefore, the usefulness of this method is evident. For instance, in the absence of any transducer coupled to the spherical detector, the mode channels model [6] does not work. This happens because the truncated icosahedron configuration is lost. However, the independent bars method still works under this condition. We are currently investigating ways to solve the inverse problem with three transducers. We already have some preliminary results that will appear in a future paper.

In this work we have actually shown that the method of the independent bars is a generalization of the mode channels model. The matrix **D** can be written in terms of the real spherical harmonics. In this case, if the transducers' positions obey the truncated icosahedron configuration, we can verify that **D** is the model matrix $\mathbf{B}_{5\times 6}$ multiplied by a constant.

Finally, we point out that this method can be used for the solution of the inverse problem in a network of bar antennas, as long as a common reference frame is chosen and time delays are accounted for.

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APPENDIX A

Each component of the matrix (11) is a function of the angles θ and ϕ that give the position of each transducer. The D functions can be written as combinations of the real spherical harmonics as follows:

$$D_{11}^L = \frac{1}{2}\left(\sqrt{\frac{16\pi}{15}}Y_1^L - \sqrt{\frac{16\pi}{45}}Y_5^L\right),$$

$$D_{12}^L = -\frac{1}{2}\sqrt{\frac{16\pi}{15}}Y_2^L, \qquad D_{13}^L = \frac{1}{2}\sqrt{\frac{16\pi}{15}}Y_3^L,$$

$$D_{22}^L = -\frac{1}{2}\left(\sqrt{\frac{16\pi}{15}}Y_1^L + \sqrt{\frac{16\pi}{45}}Y_5^L\right),$$

$$D_{23}^L = -\frac{1}{2}\sqrt{\frac{16\pi}{15}}Y_4^L, \qquad D_{33}^L = \sqrt{\frac{16\pi}{45}}Y_5^L.$$

Here the Y_m^L are defined as functions of the spherical harmonics of second order $Y_{2,n}^L$ [25]:

$$Y_1^L = \frac{i}{\sqrt{2}}(Y_{2,2}^L + Y_{2,-2}^L), \qquad Y_2^L = \frac{1}{\sqrt{2}}(Y_{2,-2}^L - Y_{2,2}^L),$$

$$Y_3^L = \frac{i}{\sqrt{2}}(Y_{2,1}^L + Y_{2,-1}^L), \qquad Y_4^L = \frac{1}{\sqrt{2}}(Y_{2,-1}^L - Y_{2,1}^L),$$

$$Y_5^L = Y_{2,0}^L.$$

Equation (10) can be written in terms of the real spherical harmonics:

$$R^L = \frac{1}{2}\sqrt{\frac{16\pi}{15}}\left[Y_1^L\left(\frac{h_{11}-h_{22}}{2}\right) + Y_2^L h_{12} + Y_3^L h_{13} + Y_4^L h_{23} + \frac{\sqrt{3}}{2}Y_5^L h_{33}\right].$$

In a more compact form this reads

$$R^L = \frac{1}{2}\sqrt{\frac{16\pi}{15}}\sum_1^5 Y_m^L h_m,$$

where the h_m are the quadrupolar representation of the gravitational wave:

$$h_1 = \left(\frac{h_{11}-h_{22}}{2}\right), \qquad h_2 = h_{12}, \qquad h_3 = h_{13},$$

$$h_4 = h_{23}, \qquad h_5 = \frac{\sqrt{3}}{2}h_{33}.$$

Therefore the matricial system of Eq. (10) takes the form

$$\begin{bmatrix} R^1 \\ R^2 \\ R^3 \\ R^4 \\ R^5 \\ R^6 \end{bmatrix} = \frac{1}{2}\sqrt{\frac{16\pi}{15}}\begin{bmatrix} Y_1^1 & Y_2^1 & Y_3^1 & Y_4^1 & Y_5^1 \\ Y_1^2 & Y_2^2 & Y_3^2 & Y_4^2 & Y_5^2 \\ Y_1^3 & Y_2^3 & Y_3^3 & Y_4^3 & Y_5^3 \\ Y_1^4 & Y_2^4 & Y_3^4 & Y_4^4 & Y_5^4 \\ Y_1^5 & Y_2^5 & Y_3^5 & Y_4^5 & Y_5^5 \\ Y_1^6 & Y_2^6 & Y_3^6 & Y_4^6 & Y_5^6 \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \end{bmatrix},$$

which is the same as the one used in Eq. (20).

APPENDIX B

We want to show in this appendix that Eq. (2) can be written in terms of the transducer's response. First we notice that the potential of the tidal force density caused by the passage of a gravitational wave is

$$\phi(\mathbf{x}, t) = \sum_{j,k} \frac{1}{2} \rho x_j \ddot{h}_{jk}(t) x_k,$$

where ρ is the mass density and x_i is the coordinate location. The term x_i represents a vector for the position so it can be written in the following form:

$$x_i = \epsilon n_i,$$

where ϵ is the amplitude of the vector and n_i represents the unit vector of the position vector. Replacing x_i in the equation, we get

$$\phi(\mathbf{x}, t) = \sum_{j,k} \frac{1}{2} \epsilon^2 \rho n_j \ddot{h}_{jk}(t) n_k,$$

which can be written in the form

$$\phi(\mathbf{x}, t) = \sum_{j,k} \frac{1}{2} \epsilon^2 \rho \frac{\partial^2}{\partial t^2} (h_{jk}(t) n_j n_k).$$

We then have the contraction of two tensors: the wave's tensor h_{ij} and a directionality tensor that characterizes the plan in which the states of polarization of the gravitational wave act. We can impose the traceless condition in $n_i n_j$, adding the term $-\frac{1}{3}\delta_{ij}$. Therefore the equation assumes the form

$$\phi(\mathbf{x}, t) = \sum_{j,k} \frac{1}{2} \epsilon^2 \rho \frac{\partial^2}{\partial t^2} \left(h_{jk}(t) \left(n_j n_k - \frac{1}{3} \delta_{jk} \right) \right),$$

where the term $(n_j n_k - \frac{1}{3}\delta_{jk})$ is the transducer tensor of Eq. (3). Therefore we can write

$$\phi(\mathbf{x}, t) = \sum_{j,k} \frac{1}{2} \epsilon^2 \rho \frac{\partial^2}{\partial t^2} (h_{jk}(t) D_{jk}),$$

or in a more summarized form,

$$\phi(\mathbf{x}, t) = \tfrac{1}{2} \epsilon^2 \rho \ddot{\mathcal{R}},$$

where $\mathcal{R}$ is the transducer response of Eq. (1). Using this equation we can write

$$\phi(\mathbf{x}, t) = \sqrt{\frac{4\pi}{15}} \epsilon^2 \rho \sum_m \ddot{h}_m(t) Y_m.$$

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