REGULAR METRIC

DEFINITION AND CHARACTERIZATION IN THE DISCRETE PLANE

(transparencies - talk)

(see references for the full text)

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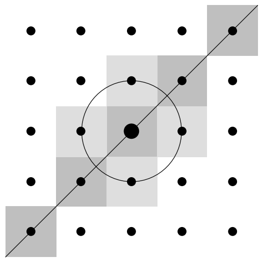
Paris, April, 2005

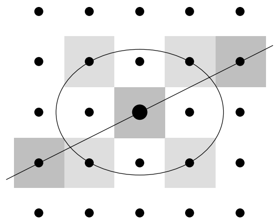
CONTENT

LOSS OF A GEOMETRICAL PROPERTY IN THE DISCRETE PLANE

(1/1)

Examples





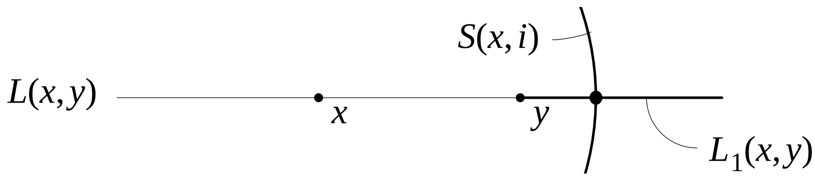
In each figure, the "circle" and the "straight–line" have no intersection in the discrete plane.

AXIOMATIC DEFINITION

(1/4)

Axiom 1

Lower regularity of type 1
(⇔ lower regularity for the triangle inequality – Kiselman, 2002)



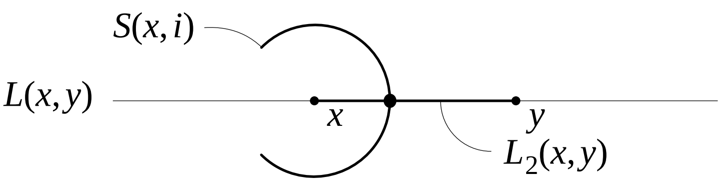
Every circle $S(x, i)$ with radius i greater than $d(x, y)$ intersects the line segment $L_1(x, y)$.

AXIOMATIC DEFINITION

(2/4)

Axiom 2

Lower regularity of type 2



Every circle $S(x, i)$ with radius i less than $d(x, y)$ intersects the line segment $L_2(x, y)$.

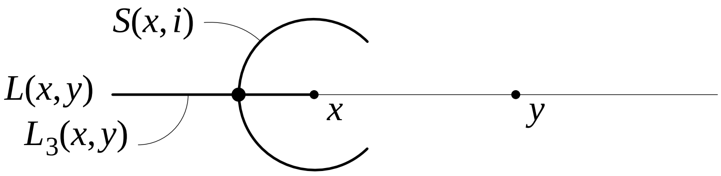
AXIOMATIC DEFINITION

(3/4)

Axiom 3

Upper regularity

(⇔ upper regularity for the triangle inequality – Kiselman, 2002)



Every circle $S(x,i)$
intersects the line segment $L_3(x,y)$.

AXIOMATIC DEFINITION
(4/4)

Definition – The metric space is (E, d) *regular* if the above three axioms are satisfied.

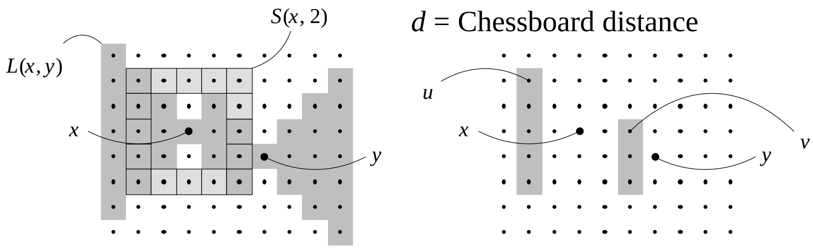
The chessboard and city block distances are regular.

EQUIVALENT DEFINITION

(1/1)

Proposition (equivalent definition) – The metric space is (E, d) *regular* iff every straight–line $L(x, y)$ and every circle $S(x, i)$ have at least two diametrically opposite points.

Example



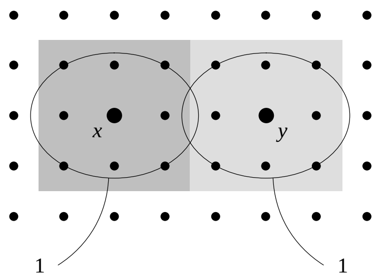
SOME PROPERTIES

(1/3)

Proposition (ball intersection) – If (E, d) is lower regular of type 2, then

$$d(x, y) \leq i + j \Rightarrow B_d(x, i) \cap B_d(y, j) \neq \emptyset.$$

Counter example 1
(elliptic distance in the discrete plane)



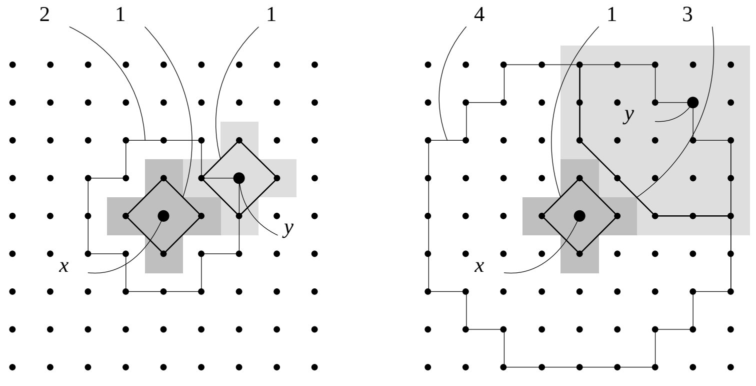
SOME PROPERTIES

(2/3)

Proposition (ball intersection) – If (E, d) is lower regular of type 2, then

$$d(x, y) \leq i + j \Rightarrow B_d(x, i) \cap B_d(y, j) \neq \emptyset.$$

Counter example 2
(octagonal distance – Rosenfeld & Pfaltz, 1968)



SOME PROPERTIES

(3/3)

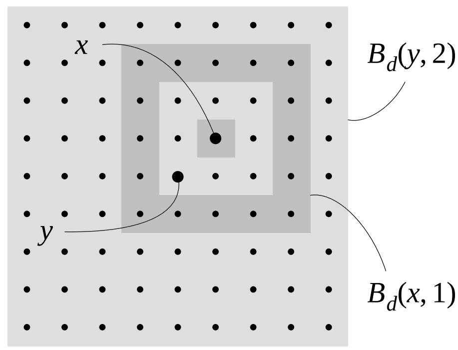
Proposition (ball inclusion) – If (E, d) is upper regular, then

$$B_d(x, i) \subset B_d(y, j) \Rightarrow d(x, y) \leq j - i.$$

Counter example

2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2
2	2	1	1	1	1	1	2	2
2	2	1	2	2	2	1	2	2
2	2	1	2	0	2	1	2	2
2	2	1	2	2	2	1	2	2
2	2	1	1	1	1	1	2	2
2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2

d



$d(x, y) = 2$

CHARACTERIZATION

(1/2)

Let B be a nonempty subset of $(\mathbf{Z}^2, +, o)$ and let j be a natural number, the *recursive Minkowski sum of B times j* is:

$$jB \triangleq \begin{cases} \{o\} & \text{if } j = 0 \\ B & \text{if } j = 1 \\ ((j-1)B) \oplus B & \text{if } 1 < j \end{cases}$$

The subset B has the *closure property* if for any natural number j:

$$jB = (jB \oplus B) \ominus B$$

Let B be a finite symmetric subset of $\mathbf{Z}^2$ such that $o \in B$ and $B \neq \{o\}$, the *induced metric* is:

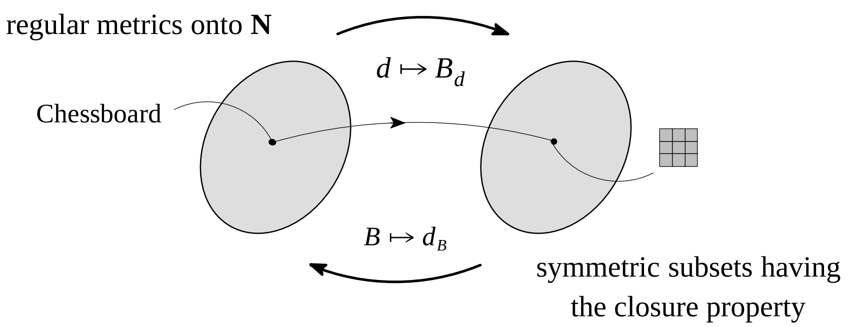
$$d_B(x,y) \triangleq \min\{j \in \mathbf{N} : x - y \in jB\}$$

Let d be a metric from $\mathbf{Z}^2 \times \mathbf{Z}^2$ onto $\mathbf{N}$, the *unit ball of d* is:

$$B_d \triangleq \{u \in \mathbf{Z}^2 : d(u,o) \leq 1\}$$

CHARACTERIZATION
(2/2)

Theorem (regular metric characterization) – The mapping $d \mapsto B_d$ from the set of translation–invariant regular metrics from $\mathbf{Z}^2 \times \mathbf{Z}^2$ onto $\mathbf{N}$ to the set of symmetric subsets of $\mathbf{Z}^2$ having the closure property is a bijection. Its inverse is $B \mapsto d_B$.



Corollary (necessary condition) – The metric spaces whose balls cannot be obtained from the unit ball through the recursive Minkowski addition are not regular (e.g. the octagonal distance).

FUTURE WORK
(1/1)

Find a sufficient condition for a subset B to have the closure property:

$$jB = (jB \oplus B) \ominus B \quad (\forall j \in \boldsymbol{N})$$

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