REGULAR METRIC

DEFINITION AND CHARACTERIZATION IN THE DISCRETE PLANE

(transparencies - talk)

(see references for the full text)

Gerald Jean Francis Banon

National Institute for Space Research – INPE
Image Processing Division – DPI
CP 515
12 245–970, São José dos Campos, SP, Brazil

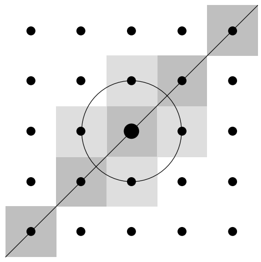
Paris, April, 2005

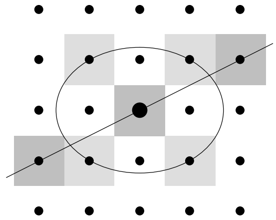
CONTENT

LOSS OF A GEOMETRICAL PROPERTY IN THE DISCRETE PLANE

(1/1)

Examples





In each figure, the "circle" and the "straight–line" have no intersection in the discrete plane.

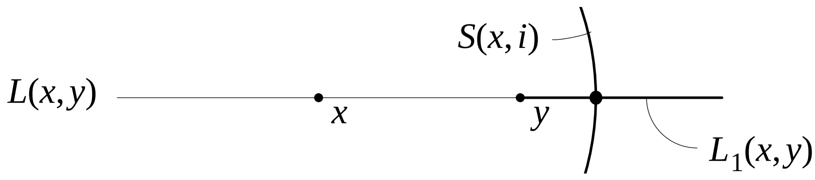
AXIOMATIC DEFINITION

(1/4)

Axiom 1

Lower regularity of type 1

(⇔ lower regularity for the triangle inequality – Kiselman, 2002)



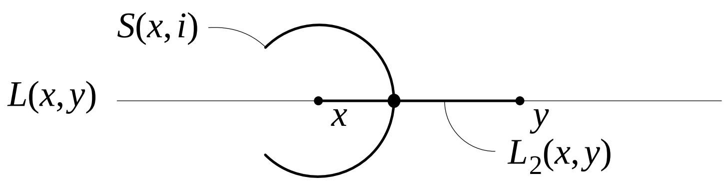
Every circle $S(x,i)$ with radius i greater than $d(x,y)$ intersects the line segment $L_1(x,y)$.

AXIOMATIC DEFINITION

(2/4)

Axiom 2

Lower regularity of type 2

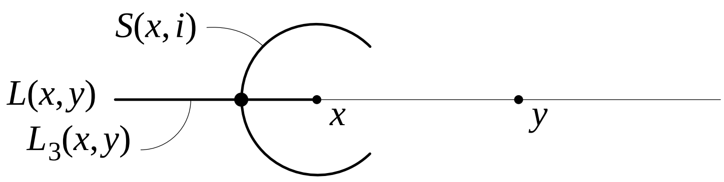


Every circle $S(x, i)$ with radius i less than $d(x, y)$ intersects the line segment $L_2(x, y)$.

AXIOMATIC DEFINITION
(3/4)

Axiom 3

Upper regularity
($\Leftrightarrow$ upper regularity for the triangle inequality – Kiselman, 2002)



Every circle $S(x,i)$
intersects the line segment $L_3(x,y)$.

AXIOMATIC DEFINITION
(4/4)

Definition – The metric space is (E, d) *regular* if the above three axioms are satisfied.

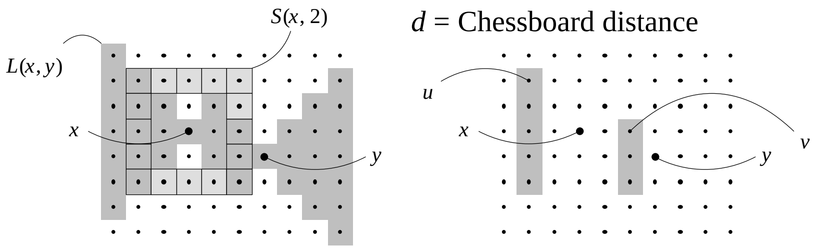
The chessboard and city block distances are regular.

EQUIVALENT DEFINITION

(1/1)

Proposition (equivalent definition) – The metric space is (E, d) *regular* iff every straight–line $L(x, y)$ and every circle $S(x, i)$ have at least two diametrically opposite points.

Example

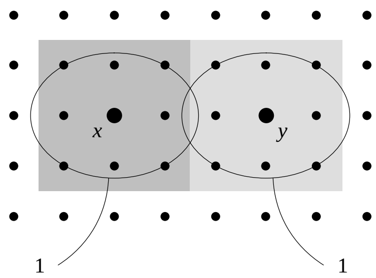


SOME PROPERTIES
(1/3)

Proposition (ball intersection) – If (E, d) is lower regular of type 2, then

$$d(x, y) \leq i + j \Rightarrow B_d(x, i) \cap B_d(y, j) \neq \emptyset.$$

Counter example 1
(elliptic distance in the discrete plane)



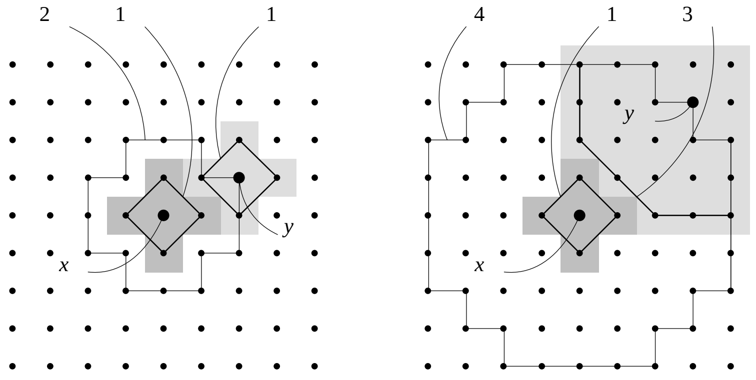
SOME PROPERTIES

(2/3)

Proposition (ball intersection) – If (E, d) is lower regular of type 2, then

$$d(x, y) \leq i + j \Rightarrow B_d(x, i) \cap B_d(y, j) \neq \emptyset.$$

Counter example 2
(octagonal distance – Rosenfeld & Pfaltz, 1968)



SOME PROPERTIES
(3/3)

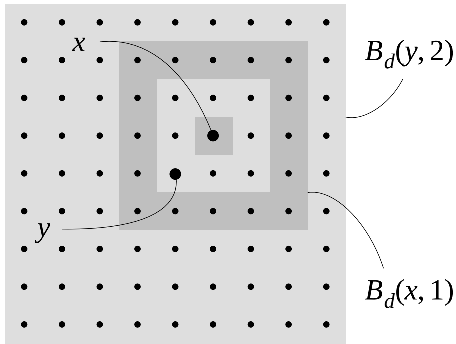
Proposition (ball inclusion) – If (E, d) is upper regular, then

$$B_d(x, i) \subset B_d(y, j) \Rightarrow d(x, y) \leq j - i.$$

Counter example

2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2
2	2	1	1	1	1	1	2	2
2	2	1	2	2	2	1	2	2
2	2	1	2	0	2	1	2	2
2	2	1	2	2	2	1	2	2
2	2	1	1	1	1	1	2	2
2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2

d



$d(x, y) = 2$

CHARACTERIZATION
(1/2)

Let B be a nonempty subset of $(\mathbf{Z}^2, +, o)$ and let j be a natural number, the *recursive Minkowski sum of B times j* is:

$$jB \triangleq \begin{cases} \{o\} & \text{if } j = 0 \\ B & \text{if } j = 1 \\ ((j-1)B) \oplus B & \text{if } 1 < j \end{cases}$$

The subset B has the *closure property* if for any natural number j:

$$jB = (jB \oplus B) \ominus B$$

Let B be a finite symmetric subset of $\mathbf{Z}^2$ such that $o \in B$ and $B \neq \{o\}$, the *induced metric* is:

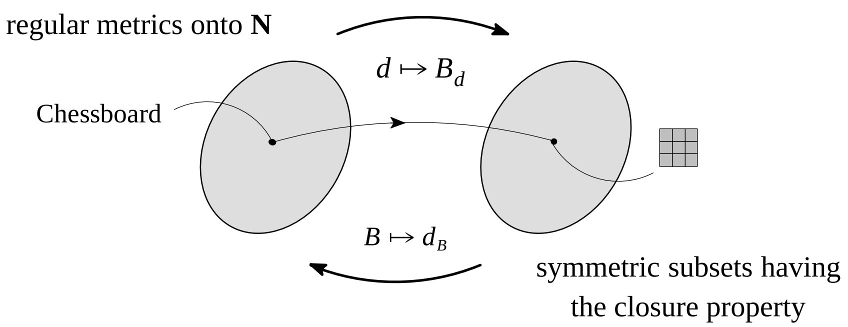
$$d_B(x,y) \triangleq \min\{j \in \mathbf{N} : x - y \in jB\}$$

Let d be a metric from $\mathbf{Z}^2 \times \mathbf{Z}^2$ onto $\mathbf{N}$, the *unit ball of d* is:

$$B_d \triangleq \{u \in \mathbf{Z}^2 : d(u,o) \leq 1\}$$

CHARACTERIZATION
(2/2)

Theorem (regular metric characterization) – The mapping $d \mapsto B_d$ from the set of translation–invariant regular metrics from $\mathbf{Z}^2 \times \mathbf{Z}^2$ onto $\mathbf{N}$ to the set of symmetric subsets of $\mathbf{Z}^2$ having the closure property is a bijection. Its inverse is $B \mapsto d_B$.



Corollary (necessary condition) – The metric spaces whose balls cannot be obtained from the unit ball through the recursive Minkowski addition are not regular (e.g. the octagonal distance).

FUTURE WORK

(1/1)

Find a sufficient condition for a subset B to have the closure property:

$$jB = (jB \oplus B) \ominus B \quad (\forall j \in \boldsymbol{N})$$

REFERENCES

Banon, Gerald J. F. Regular metric: definition and characterization in the discrete plane. In: International Symposium on Mathematical Morphology and its Applications to Image and Signal Processing, 7., 18–20 April 2005, Paris, France. **Proceedings**... 2005. Published as: INPE–12172–PRE/7511.

Banon, G. J. F. **Regular metric**: definition and characterization in the discrete plane. São José dos Campos: Instituto Nacional de Pesquisas Espaciais, 2004. (INPE ePrint sid.inpe.br/yolanda/2004/09.08.16.30). URL*ib* repository: <sid.inpe.br/yolanda/2004/09.08.16.30>.

Banon, Gerald J. F. **Characterization of the integer–valued translation–invariant regular metrics on the discrete plane**. São José dos Campos: Instituto Nacional de Pesquisas Espaciais (INPE), 2004. 26 p. (INPE–11277–NTC/363). URL*ib* repository: <dpi.inpe.br/banon/2004/03.08.12.12>.

Kiselman, Christer O. **Digital geometry and mathematical morphology**. Lecture Notes, Spring Semester, Uppsala University, 2002. Posted at <http://www.math.uu.sc/kiselman/dgmm2002.pdf>.

Rosenfeld, Azriel and Pfaltz, John L. Distance functions on digital pictures. **Pattern Recognition**, v. 1, n. 1, pp. 33–61, 1968.