1. Introduction

Consider a gyroscope consisting of a rotor and two gimbals (see Fig. 1).

The external gimbal is connected with an external case at points $A_1$ and $A_2$ by a pair of axes. At points $B_1$ and $B_2$ the external gimbal is connected with the internal one by a second pair of axes.
Finally, the rotor is connected at points C₁ and C₂ with the internal gimbal by a third pair of axes.

All axes allow free rotation.

The aim of this work is to establish the formulation of equations expressing the influence of

(i) gimbal motion (gimbals not considered massless), and
(ii) air drag

on the rotor motion.

Dyadic calculus is consistently used during the derivation of equations, [1].

2) GYROSCOPE POSITIONING AND MOVEMENT

To establish the gyroscope position during its movement, several coordinate systems will be chosen:

S₀ - The origin coincides with the center of mass of the gyroscope. z₀ axis coincides with A₁ A₂, x₀ and y₀ are chosen arbitrarily, fixed relative to the case.

Sₐ - Fixed on the external gimbal, with zₐ = z₀ and yₐ coinciding with B₁ B₂.

Sₜ - Fixed on the internal gimbal, with yₜ = yₐ and zₜ coinciding with C₁ C₂.

Sₐ - Fixed on the rotor: zₐ = z₀; xₐ and yₐ in the rotor plane.

The gyroscope can be conducted from its position of reference, S₀, to its arbitrary position, Sₐ, by means of:
(i) Rotation (of the external gimbal) through angle $\phi$, around $A_1 A_2$.

(ii) Rotation (of the internal gimbal) through angle $\Theta$, around $B_1 B_2$.

(iii) Rotation (of the rotor) through $\psi$ around $C_1 C_2$.

The angles $\phi$, $\Theta$, $\psi$ (known as Euler Angles) determine completely the gyroscope position at any instant. Their rates $\dot{\phi}$, $\dot{\Theta}$, $\dot{\psi}$ determine, respectively, angular velocities of

(i) precession

(ii) nutation

(iii) spin

Fig. 2 - $\phi$, $\Theta$ and $\psi$. 
3) COORDINATE SYSTEM TRANSFORMATIONS

Observe that transformation $S_0 \rightarrow S_B$ consists of

(i) rotation through $\phi$ around $\hat{z}_0$

(ii) rotation through $\theta$ around the new position of $\hat{y} = \hat{y}_A$

This way, the transformation matrix $\mathbf{L}_{BA}$ (i.e. from $S_A$ to $S_B$) is

$$\mathbf{L}_{BA} = \begin{pmatrix}
\cos\theta & 0 & -\sin\theta \\
0 & 1 & 0 \\
\sin\theta & 0 & \cos\theta
\end{pmatrix} \begin{pmatrix}
\cos\phi & \sin\phi & 0 \\
-\sin\phi & \cos\phi & 0 \\
0 & 0 & 1
\end{pmatrix}$$

$$= \begin{pmatrix}
\cos\phi \cos \theta & \cos\phi \sin \theta & -\sin \theta \\
-\sin \phi & \cos \phi & 0 \\
\sin \phi \cos \theta & \sin \phi \sin \theta & \cos \theta
\end{pmatrix}$$

In particular

$$\hat{z}_0 = (-\sin\theta) \hat{x}_B + (\cos\theta) \hat{z}_B$$

Transformation matrix $\mathbf{L}_{RB}$ (i.e. from $S_B$ to $S_R$) is

$$\mathbf{L}_{RB} = \begin{pmatrix}
\cos \psi & \sin \psi & 0 \\
-\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{pmatrix}$$

This way

$$\begin{pmatrix}
\hat{x}_R \\
\hat{y}_R \\
\hat{z}_R
\end{pmatrix} = \begin{pmatrix}
\cos \psi & \sin \psi & 0 \\
-\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
\hat{x}_B \\
\hat{y}_B \\
\hat{z}_B
\end{pmatrix} = \begin{pmatrix}
\hat{x}_B \cos \psi + \hat{y}_B \sin \psi \\
-\hat{x}_B \sin \psi + \hat{y}_B \cos \psi \\
\hat{z}_B
\end{pmatrix}$$
4) ANGULAR VELOCITIES

The angular velocity of $S_B$ relative to the case (i.e. $S_A$) is

$$\omega_B^O = \phi \ddot{z}_A + \dot{\theta} \dot{y}_B$$  \hspace{1cm} (3)

In particular, substituting (2) into (3), one gets

$$\omega_B^O = -(\sin \theta) \phi \ddot{z}_B + \dot{\theta} \dot{y}_B + (\cos \theta) \dot{\phi} \dot{z}_B.$$  \hspace{1cm} (4)

The angular velocity of the rotor relative to $S_B$ is

$$\omega_R^B = \psi \dot{z}_B$$

This way one gets

$$\omega_R^O = \omega_R^B + \omega_B^A = -(\phi \sin \theta) \ddot{x}_B + \dot{\theta} \dot{y}_B + (\psi + \phi \cos \theta) \dot{z}_B$$  \hspace{1cm} (5)

5) MOMENTS OF INERTIA

The dyadic of inertia of the three parts are:

Rotor

$$J_R = I_{R}^{x} \dddot{x}_R \dddot{x}_R + I_{R}^{y} \dddot{y}_R \dddot{y}_R + I_{R}^{z} \dddot{z}_R \dddot{z}_R$$

$$= I_{R}^{x} (\dddot{x}_B \cos \psi + \dddot{y}_B \sin \psi)(\dddot{z}_B \cos \psi - \dddot{y}_B \sin \psi) +$$

$$+ I_{R}^{x} (-\dddot{x}_B \sin \psi + \dddot{y}_B \cos \psi)(\dddot{x}_B \sin \psi + \dddot{y}_B \cos \psi) +$$

$$+ I_{R}^{x} \dddot{z}_B \dddot{z}_B$$

$$= I_{R}^{x} \dddot{x}_B \dddot{x}_B + I_{R}^{y} \dddot{y}_B \dddot{y}_B + I_{R}^{z} \dddot{z}_B \dddot{z}_B$$  \hspace{1cm} (6)
6) ANGULAR MOMENTUM

One has

\[ \mathbf{\dot{\mathbf{n}}} = \int \mathbf{\dot{\mathbf{R}}} \times \mathbf{\dot{\mathbf{R}}} \, dm = \sum_i \left( \int \mathbf{\dot{R}}^i \times \mathbf{\dot{R}}^i \, dm \right) = \sum_i \mathbf{\dot{\mathbf{n}}}^i \]

This means that the contributions of all parts are additive.

The contribution of

Rotor

\[ \mathbf{\dot{\mathbf{n}}}^R = \mathbf{J}_R^* \mathbf{\hat{\omega}}^R = (I_R^* \dot{\mathbf{x}}_B^i \mathbf{\hat{x}}_B^i + I_R^* \dot{\mathbf{y}}_B^i \mathbf{\hat{y}}_B^i + I_R^* \dot{\mathbf{z}}_B^i \mathbf{\hat{z}}_B^i) \cdot \left[ -\dot{\mathbf{x}}_B^i \phi \sin \Theta + \dot{\mathbf{y}}_B^i \Theta + \dot{\mathbf{z}}_B^i (\dot{\psi} + \dot{\phi} \cos \Theta) \right] \]

\[ = \mathbf{\dot{x}}_B^i (-I_R^* \phi \sin \Theta) + \mathbf{\dot{y}}_B^i (I_R^* \Theta) + \mathbf{\dot{z}}_B^i [I_R^* (\dot{\psi} + \dot{\phi} \cos \Theta)] \]

(9)
Internal gimbal

\[ \dot{h}_B = J_B \cdot \ddot{\omega}^B/0 = -\ddot{x}_B \cdot I_B^+ \cdot \phi \sin \theta + \ddot{y}_B \cdot I_B^+ \cdot \phi + z_B \cdot I_B^+ \cdot \phi \cos \theta \]  

(10)

External gimbal

\[ \dot{h}_A = J_A \cdot \ddot{\omega}^A/0 = (I_A^+ \ddot{x}_A^+ + \ddot{y}_A^+ + \ddot{z}_A^+) \cdot (\phi \ddot{z}_A) \]

\[ = I_A^{++} \phi \dot{z}_A - \ddot{x}_B \cdot I_A^{++} \phi \sin \theta + z_B \cdot I_A^{++} \phi \cos \theta \]  

(11)

This way, the total angular momentum can be written

\[ \dot{h} = P \ddot{x}_B + Q \ddot{y}_B + R \ddot{z}_B \]  

(12)

where

\[
\begin{cases}
P = -I_\alpha \phi \sin \theta \\
Q = I_\beta \phi \\
R = I_\gamma \phi \cos \theta
\end{cases}
\]

and where

\[
\begin{cases}
I_\alpha = I^{++} + I^{+} + I^- \\
I_\beta = I^{++} + I^{+} + I^- \\
I_\gamma = I^{++} + I^{+} + I^- \\
\end{cases}
\]

7) EQUATIONS OF MOVEMENT (EULER EQUATIONS)

As is well known, one has

\[ \dot{h} = G \]

A more convenient way is to express \( h \) in the inertial system, \( \ddot{h} \) then no \( \omega \times h \) term will be required and equations of motion are directly integrable.

where

\[ \dot{h} = \text{total angular momentum} \]

\[ \ddot{h} = \text{resultant of external torques} \]
and where the symbol \( (\cdot') \) denotes the inertial time derivative \( (d/dt) \).

Expressing equations in a non-inertial system, say \( S_B \), one has:

\[
\dot{\mathbf{h}} = \dot{\mathbf{h}}_B + \omega^{B/0} \times \mathbf{h}
\]  

(14)

where the symbol \( (\cdot)'_B \) denotes the time derivative as determined by the observer in \( S_B \), that is—see eq. (12):

\[
\dot{\mathbf{h}}_B = \dot{\mathbf{x}}_B + \mathbf{Q} \times \mathbf{y}_B + \mathbf{R} \times \mathbf{z}_B
\]

Even though this is a standard procedure of getting Euler equations, resulting in:

\[
\begin{align*}
\dot{P} + \mathbf{R}^\phi - \mathbf{Q}^\phi \cos\theta & = G_x^B \\
\dot{Q} + \mathbf{P}^\phi \cos\theta + \mathbf{R}^\phi \sin\theta & = G_y^B \\
\dot{R} + \mathbf{P}^\phi - \mathbf{Q}^\phi \sin\theta & = G_z^B
\end{align*}
\]

(15)

or, after some algebra, in

\[
\begin{align*}
-I^\alpha \phi \sin\theta + (I^\gamma - I^\alpha - I^\beta) \hat{\phi} \overline{\hat{\theta}} \cos\theta & + I^\gamma \hat{\psi} \overline{\hat{\theta}} = G_x^B \\
I^\beta \hat{\phi} + (I^\alpha - I^\beta - I^\gamma) \phi^2 \cos\theta \sin\theta + I^\gamma \phi \sin\theta & = G_y^B \\
I^\gamma \phi \cos\theta + (I^\alpha - I^\beta - I^\gamma) \overline{\phi} \theta \sin\theta & = G_z^B
\end{align*}
\]

one gets, this way, second order equations which are not immediately integrable.

A more convenient way is to express \( \mathbf{h} \) in the inertial system \( S_0 \), then no \( \mathbf{\Omega} \times \mathbf{h} \) term will be required and equations of motion will be directly integrable.

Of course, one has

\[
(\mathbf{h})_0 = \mathbf{U}^0_B (\mathbf{h})_B
\]

(18)
where \((\mathbf{h})_0\) and \((\mathbf{h})_B\) express column matrices composed of the components of angular momentum \(J\) and \(S_B\), respectively.

Thus, see eq. (1)

\[
(\mathbf{h})_0 = \begin{bmatrix} \cos \theta \cos \phi & - \sin \phi \\ \cos \theta \sin \phi & \cos \phi \\ - \sin \theta & 0 \end{bmatrix} \begin{bmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{bmatrix} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}
\]

After some algebra, Euler equations can be written as

\[
\frac{d}{dt} \left\{ (I_Y - I_\alpha) \hat{\phi} \cos \theta \sin \phi - I_B \hat{\theta} \sin \phi + I_R \hat{\psi} \sin \phi \cos \phi \right\} = G_{x_0}
\]

\[
\frac{d}{dt} \left\{ (I_Y - I_\alpha) \hat{\phi} \cos \theta \sin \phi + I_B \hat{\phi} \sin \phi + I_R \hat{\psi} \sin \phi \sin \phi \right\} = G_{y_0}
\]

\[
\frac{d}{dt} \left\{ \hat{\phi} (I_\alpha \sin^2 \Theta + I_Y \cos^2 \Theta) + I_R \hat{\psi} \sin \phi \right\} = G_{z_0}
\]

The above equations can be, now, immediately integrated and, since they will result in a system of linear equation in \(\hat{\phi}, \hat{\theta}, \hat{\psi}\), can also be solved with respect of these variable.

8) TIME RATES OF EULER ANGLES WITH EXTERNAL TORQUES ABSENT

Assuming \(G_{x_0} = G_{y_0} = G_{z_0} = 0\), integrating the above equations and solving them for \(\hat{\phi}, \hat{\theta}, \hat{\psi}\), one gets, after some manipulations.

\[
\hat{\phi} = \frac{1}{I_\alpha} \left( c_1 \sin \phi + c_2 \cos \phi \right) \\
\hat{\theta} = \frac{1}{I_B} \left( c_3 \sin \theta - (c_1 \cos \phi + c_2 \sin \phi) \cos \theta \right) \\
\hat{\psi} = \frac{1}{I_R \sin \theta} \left( c_1 \cos \phi + c_2 \sin \phi \right) (\sin^2 \Theta + I_Y \cos^2 \Theta) + \frac{I_\alpha - I_Y}{I_\alpha} \sin \phi \cos \phi
\]

(18)
9) AIR DRAG DUE TO THE ROTATING DISK

Rotating disk produces a moment due to the air drag in \( \hat{Z}_R \) direction.

To estimate the drag in turbulent flow (which is our case), choose 1/7 power for velocity distribution.

Centrifugal force per unit volume is \( \rho \, r \, \omega^2 \) \( (\omega = \text{angular velocity}) \) and the centrifugal force acting on a volume \( dr \times ds \times \delta \) \( (\delta = \text{boundary layer thickness}) \) becomes \( \rho r \omega^2 \, ds \, dr \, \delta \). The shearing stress \( \tau_0 \) forms an angle \( \theta \) with the tangential direction and its radial component must balance the centrifugal force. Hence

\[
\tau_0 \, \sin \theta \, dr \, ds = \rho r \omega^2 \, \delta \, dr \, ds
\]

This result shows good agreement for \( \text{Rey} > 3 \times 10^5 \).

Or

\[
\tau_0 \, \sin \theta = \rho r \omega^2 \delta
\]

Using analogy with flat plate, one has

\[
\tau_0 \, \cos \theta = \rho (\omega r)^{7/4} \frac{(v/\delta)^{1/4}}{\text{Re}^{1/6}}
\]

\( (U_\infty \text{ substituted by } r \omega) \). Then

\[
\delta = r^{3/5} \frac{(v/\omega)^{1/5}}{\text{Re}^{1/6}}
\]

The torque becomes

\[
M = \tau_0 \, R^3 = \rho R \omega^2 \left( \frac{v}{\omega} \right)^{1/5} \left( \frac{R}{\delta} \right)^{3/5} R^3
\]
\[ \mathbf{M} = \rho U^2 R^3 \frac{(v)}{UR}^{1/5} \]

where

\[ U = \omega R. \]

Von Karman using the 1/7 power law for the variation of the tangential velocity component through the boundary layer showed that, for a disk wetted on both sides, the viscous torque is equal to

\[ 2M = 0.073 \rho \omega^2 R^5 \left( \frac{v}{\omega R^2} \right)^{1/5} \tag{19} \]

Thus, \( c_M \) becomes

\[ c_M = \frac{0.146}{(\text{Rey})^{1/5}} \]

This result shows good agreement for \( \text{Rey} > 3 \times 10^5 \).

REFERENCES:

Fl vs TIME

kx=lb=169, lx=lr=330.
FI-DOT VS TIME

(giro02)

A = B = C = D = 0
THETA / THETA-DOT

(gh02)

-- A = B = C = D = 0
PSI vs TIME

giro02

Time

A = B = C = D = 0
PSI-DOT VS TIME

(giro02)

A = B = C = D = 0
THETA-DOT vs TIME

(time)

--- A = B = C = D = 0